On Parabolic Stochastic Integro-Differential Equations

Existence, Regularity, and Numerics

THE UNIVERSITY *of* EDINBURGH

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Doctor of Philosophy
University of Edinburgh
June 2015

Declaration

I hereby declare that this thesis was composed by myself and that the work contained herein is my own, except where explicitly stated otherwise. Further, I declare that this work has not been submitted for any other degree or professional qualification.

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June 2015

Abstract

In this thesis, we study the existence, uniqueness, and regularity of systems of degenerate linear stochastic integro-differential equations (SIDEs) of parabolic type with adapted coefficients in the whole space. We also investigate explicit and implicit finite difference schemes for SIDEs with non-degenerate diffusion. The class of equations we consider arise in non-linear filtering of semimartingales with jumps.

In Chapter 2, we derive moment estimates and a strong limit theorem for space inverses of stochastic flows generated by Lévy driven stochastic differential equations (SDEs) with adapted coefficients in weighted Hölder norms using the Sobolev embedding theorem and the change of variable formula. As an application of some basic properties of flows of Weiner driven SDEs, we prove the existence and uniqueness of classical solutions of linear parabolic second order stochastic partial differential equations (SPDEs) by partitioning the time interval and passing to the limit. The methods we use allow us to improve on previously known results in the continuous case and to derive new ones in the jump case. Chapter 3 is dedicated to the proof of existence and uniqueness of classical solutions of degenerate SIDEs using the method of stochastic characteristics. More precisely, we use Feynman-Kac transformations, conditioning, and the interlacing of space inverses of stochastic flows generated by SDEs with jumps to construct solutions.

In Chapter 4, we prove the existence and uniqueness of solutions of degenerate linear stochastic evolution equations driven by jump processes in a Hilbert scale using the variational framework of stochastic evolution equations and the method of vanishing viscosity. As an application, we establish the existence and uniqueness of solutions of degenerate linear stochastic integro-differential equations in the L^2-Sobolev scale.

Finite difference schemes for non-degenerate SIDEs are considered in Chapter 5. Specifically, we study the rate of convergence of an explicit and an implicit-explicit finite difference scheme for linear SIDEs and show that the rate is of order one in space and order one-half in time.

Key words: Stochastic flows, stochastic differential equations (SDEs), Lévy processes, strong-limit theorem, stochastic partial differential equations (SPDEs), degenerate parabolic type, parabolic stochastic integro-differential equations (SIDEs), partial integro-differential equations (PIDEs), non-local operators, method of stochastic characteristics, Ito-Wentzell formula, stochastic evolution equations, vanishing viscosity, finite difference schemes

AMS subject classifications (MSC2010). 60H10, 60J75, 60F15, 60H15, 34F05, 35F40, 35R60, 35K15, 35K65, 60H20, 45K05, 65M06, 65M12

Lay Summary

In this thesis, we investigate a class of equations governing random processes that depend on time and space. These equations are a generalization of deterministic parabolic partial differential equations (PDEs). The prototypical example of a parabolic PDE is the heat equation, which is an equation that governs the evolution of heat in some medium in time and space. Randomness can be introduced to the heat equation through a random source and sink of heat at certain points of time and space, a random initial heat profile, and a random conductivity coefficient of the medium. The heat equation is an example of a diffusion equation, which is an equation that describes the density of particles as they diffuse according to some law. The physical law that governs the heat equation is called Fick's law and it describes only a special type of diffusion. The equations that we study, even if we take them to be deterministic, allow for a much wider range of diffusions than the prototypical heat equation.

When investigating an equation, a natural first question to ask is whether there exists a solution to the equation in some well-defined sense. Moreover, if there exists a solution, then one ought to ask whether it is unique, and if so, what properties it has. The equations we consider have range of possible inputs, and thus we want to know how the properties of the solution depend on the inputs. There are many properties of the equations to explore, but we consider only a property that is referred to as regularity. Regularity is characterized by a class of spaces to which the solution and inputs belong that are ordered by inclusion; smaller spaces correspond to higher regularity. It is natural to expect that if you take more regular inputs, then the solution should be more regular. This is the case for the equations we consider.

It is rare for the solution of an equation we study to have a closed form expression of time, space, and randomness. Nevertheless, it is still possible to prove that there exists a unique solution and to study the regularity of that solution. In fact, there are different approaches to accomplish this objective and each way has its own advantage. We will discuss two such approaches in this thesis. Since we can not expect a closed form expression for the solutions, it also practical and interesting to develop some method to approximate the solutions and to prove that the approximate solutions are close in some well-defined sense to the true solution that we know exists. In the final chapter, we describe a simple approximation scheme for a special subclass of the equations considered. The approximate solutions are defined on some countable set of points in time and space and satisfy an equation that can be solved by simple algebraic manipulations for some realization of the random variables in the equation.

I dedicate this thesis to my family.

Acknowledgements

First, I would like to thank my supervisor, István Gyöngy, for the support and guidance he has provided me during my time at the University of Edinburgh. I am indebted to Remigijus Mikulevičius for the opportunity to work with him at the University of Southern California in the Fall of 2013 and to collaborate with him on numerous projects. I have also benefited from having Kontantinos Dareiotis as a friend, flatmate, and collaborator.

The research contained in this thesis would not have been possible without funding from the Principal's Career Development Scholarship and the Edinburgh Global Research Scholarship and support from Gill Law, Graduate School Administrator.

I have been fortunate enough to participate in numerous discussions with other students that have directly contributed to my academic development. In particular, I would like to mention: Eric Hall, Máté Gerencsèr, Brian Hamilton, Marc Ryser, David Cottrell, Andrea Meireles Rodrigues, Sebastian Vollmer, Chaman Kumar, and José Rodríguez Villarreal. In fact, it was Marc Ryser who introduced me to Stochastic Partial Differential Equations and suggested I apply to the University of Edinburgh to work with István Gyöngy.

I would like to thank my friends and family, especially my mother, grandparents, and Aunt Mary Ann and Uncle Greg, for supporting all my endeavours. Last, I am grateful for the love, patience, and encouragement Bihotz Barrenetxea Dominguez has shown me.

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Chapter 1
Introduction

The subject of this thesis is parabolic linear stochastic integro-differential equations (SIDEs). These equations arise in the study of non-linear filtering of semimartingales, scaled limits of particle systems, flows of stochastic diffeomorphisms, and mathematical biology, physics, and finance. For the author, the strongest impetus to the advancement of the theory of SIDEs is the problem of non-linear filtering of jump diffusions.

The first derivation of the filtering equations for semimartingales with jumps is due to B. Grigelionis in [Gri72]. The reduced form filtering equations, which are linear SIDEs, were derived in [Gri76]. A proof of existence and uniqueness of solutions to the reduced form filtering equations was given in [Tin77b] by E. Tinfavičius under the assumption of non-degenerate stochastic parabolicity. While in [Tin77b] essentially the variational approach of SPDEs was used, one important aspect of this approach was missing, namely Itô's formula for the square of the norm for jump processes. Such a result is used to conclude that the variational solution of a stochastic evolution equation is càdlàg with values in the pivot space and to get an energy estimate of the supremum in time of the norm of the solution in the pivot space. In [GK81], I. Gyöngy and N.V. Krylov derived Itô's formula for the square of the norm in the jump case, and the corresponding existence and uniqueness result for monotone stochastic evolution equations with jumps was studied in [Gyö82]. In [Gri82], B. Grigelionis applied the result in [GK81] to complete the variational existence and uniqueness result for the reduced form equations in the non-degenerate setting that began in [Tin77b]. The reader that is interested in the derivation of the reduced form filtering equations for semimartingales with jumps is advised to consult the article [Gri82] and the review article [GM11].

It is also worth mentioning that the stochastic evolution equations driven by additive càdlàg martingale noise were studied by G. Pistone and M. Métivier in Section 5 of [MP76] using the semigroup approach. One important aspect of the reduced form filtering equation is that, in general, the principal part of the operators in the drift of these equations depend on space and time and are random. The standard semigroup approach to stochastic equations in infinite dimensions cannot treat these type of equations, since if the semigroup generated by the principal part is random, the stochastic convolution does not make sense as an ordinary Itô integral. In a recent work, M. Pronk and M. Veraar [PV14] showed that it is possible to extend the semigroup approach to treat equations where the principal part of stochastic evolution equation is random. However, at the time of writing, there still seems to be some limitations in this approach; for example, some regularity in time is needed for the diffusion coefficient in the stochastic heat equation when applying their theory. We emphasize that this is not needed in the variational approach.

This thesis is dedicated to the the proof of existence, uniqueness, and regularity of fully degenerate linear SIDEs. Specifically, we derive a theory for these equations in Hölder spaces using the method of stochastic characteristics and a theory in L^2-Sobolev spaces using the variational approach of stochastic evolution equations and the method of vanishing viscosity. We also investigate finite difference approximations of SIDEs with non-degenerate diffusion.

Let us state the general form of the equation that we will investigate in this thesis. Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a complete filtered probability space satisfying the usual conditions of right-continuity and completeness. For each real number $T > 0$, we let $\mathcal{R}_T$ and $\mathcal{P}_T$ be the $\mathbf{F}$-progressive and $\mathbf{F}$-predictable sigma-algebra on $\Omega \times [0, T]$, respectively. For our driving processes, we take a sequence w^{ϱ}_t, $t \geq 0$, $\varrho \in \mathbf{N}$, of independent one-dimensional $\mathbf{F}$-adapted Wiener processes and a $\mathbf{F}$-adapted Poisson random measure $p(dt, dz)$ on $(\mathbf{R}_+ \times Z_1, \mathcal{B}(\mathbf{R}_+) \otimes \mathcal{Z}_1)$ with intensity measure $\pi_1(dz)dt$, where $(Z_1, \mathcal{Z}_1, \pi_1)$ is a sigma-finite measure space. Denote by $q(dt, dz) = p(dt, dz) - \pi_1(dz)dt$ the compensated Poisson random measure. Let $D^1, E^1, V^1 \in Z$ be disjoint $\mathcal{Z}^1$-measurable subsets such that $D^1 \cup E^1 \cup V^1 = Z^1$ and $\pi(V^1) < \infty$. Let $(Z^2, \mathcal{Z}^2, \pi^2)$ be a sigma-finite measure space and $D^2, E^2 \in Z^2$ be disjoint $\mathcal{Z}^2$-measurable subsets such that $D^2 \cup E^2 = Z^2$.

Fix an arbitrary positive real number $T > 0$ and integers $d_1, d_2 \geq 1$. Let $\alpha \in (0, 2]$ and let $\varphi : \Omega \times \mathbf{R}^{d_1} \to \mathbf{R}^{d_2}$ be $\mathcal{F}_0 \otimes \mathcal{B}(\mathbf{R}^{d_1})$-measurable. We consider the system of SIDEs on $[0, T] \times \mathbf{R}^{d_1}$ given by

$$\begin{aligned} du^l_t &= \left((\mathcal{L}^{1;l}_t + \mathcal{L}^{2;l}_t)u_t + \mathbf{1}_{[1,2]}(\alpha) b^{i;l\bar{l}}_t \partial_i u^{\bar{l}}_t + c^{l\bar{l}}_t u^{\bar{l}}_t + f^l_t\right) dt + \left(\mathcal{N}^{l\varrho}_t u_t + g^{l\varrho}_t\right) dw^{\varrho}_t \\ &\quad + \int_{Z^1} \left(\mathcal{I}^{1;l}_{t,z} u_{t-} + h^l_t(z)\right) [\mathbf{1}_{D^1}(z) q(dt, dz) + \mathbf{1}_{E^1 \cup V^1}(z) p(dt, dz)], \quad t \leq T, \\ u^l_0 &= \varphi^l, \quad l \in \{1, \ldots, d_2\}, \end{aligned} \qquad \textbf{SIDE}$$

where for $\phi \in C^{\infty}_c(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$, $k \in \{1, 2\}$, and $l \in \{1, \ldots, d_2\}$,

$$\begin{aligned} \mathcal{L}^{k;l}_t \phi(x) :&= \mathbf{1}_{\{2\}}(\alpha) a^{k;ij}_t(x) \partial_{ij} \phi^l(x) + \int_{D^k} \rho^{k;l\bar{l}}_t(x, z) \left(\phi^{\bar{l}}(x + H^k_t(x, z)) - \phi^{\bar{l}}(x)\right) \pi^k(dz) \\ &\quad + \int_{D^k} \left(\phi^l(x + H^k_t(x, z)) - \phi^l(x) - \mathbf{1}_{(1,2]}(\alpha) H^{k;i}_t(x, z) \partial_i \phi^l(x)\right) \pi^k(dz) \\ &\quad + \mathbf{1}_{\{2\}}(k) \int_{E^2} \left((I^{l\bar{l}}_{d_2} + \rho^{2;l\bar{l}}_t(x, z)) \phi^{\bar{l}}(x + H^2_t(x, z)) - \phi^l(x)\right) \pi^2(dz), \\ \mathcal{N}^{l\varrho}_t \phi(x) :&= \mathbf{1}_{\{2\}}(\alpha) \sigma^{i\varrho}_t(x) \partial_i \phi^l(x) + \upsilon^{l\bar{l}\varrho}_t(x) \phi^{\bar{l}}(x), \quad \varrho \in \mathbf{N}, \end{aligned}$$

$$\mathcal{I}^l_{t,z}\phi(x) := (I^{l\bar{l}}_{d_2} + \rho^{1;l\bar{l}}_t(x,z))\phi^{\bar{l}}(x + H^1_t(x,z)) - \phi^l(x),$$

and

$$\int_{D^k} \left(|H^k_t(x,z)|^\alpha + |\rho^k_t(x,z)|^2\right)\pi^k(dz) + \int_{E^k} \left(|H^k_t(x,z)|^{1\wedge\alpha} + |\rho^k_t(x,z)|\right)\pi^k(dz) < \infty.$$

The summation convention with respect to repeated indices $i, j \in \{1,\ldots,d_1\}$, $\bar{l} \in \{1,\ldots,d_2\}$, and $\varrho \in \mathbf{N}$ is used here and below. The $d_2 \times d_2$ dimensional identity matrix is denoted by I_{d_2}. For a subset A of a larger set X, $\mathbf{1}_A$ denotes the $\{0,1\}$-valued function taking the value 1 on the set A and 0 on the complement of A. We assume that for each $k \in \{1,2\}$,

$$\sigma^k_t(x) = (\sigma^{k;i\varrho}_t(\omega,x))_{1\le i\le d_1,\,\varrho\in\mathbf{N}},\quad b_t(x) = (b^i_t(\omega,x))_{1\le i\le d_1},\quad c_t(x) = (c^{l\bar{l}}_t(\omega,x))_{1\le l,\bar{l}\le d_2},$$

$$\upsilon^k_t(x) = (\upsilon^{k;l\bar{l}\varrho}_t(\omega,x))_{1\le l,\bar{l}\le d_2,\,\varrho\in\mathbf{N}},\quad f_t(x) = (f^i_t(\omega,x))_{1\le i\le d_2},\quad g_t(x) = (g^{i\varrho}_t(\omega,x))_{1\le i\le d_2,\,\varrho\in\mathbf{N}},$$

are random fields on $\Omega\times[0,T]\times\mathbf{R}^{d_1}$ that are $\mathcal{R}_T\otimes\mathcal{B}(\mathbf{R}^{d_1})$-measurable. For each $k \in \{1,2\}$, we assume that

$$H^k_t(x,z) = (H^{k;i}_t(\omega,x,z))_{1\le i\le d_1},\ \rho^k_t(x,z) = (\rho^{k;l\bar{l}}_t(\omega,x,z))_{1\le l,\bar{l}\le d_2},$$

are random fields on $\Omega\times[0,T]\times\mathbf{R}^{d_1}\times Z^k$ that are $\mathcal{P}_T\otimes\mathcal{B}(\mathbf{R}^{d_1})\otimes\mathcal{Z}^k$-measurable. Moreover, we assume that $h_t(x,z) = (h^i_t(\omega,x,z))_{1\le i\le d_2}$ is a random field on $\Omega\times[0,T]\times\mathbf{R}^{d_1}\times Z^1$ that is $\mathcal{P}_T\otimes\mathcal{B}(\mathbf{R}^{d_1})\otimes\mathcal{Z}^1$-measurable.

This thesis is organized as follows. Chapter 2 is dedicated to establishing some properties of space inverses of stochastic flows generated by an SDEs with jumps. These properties play an important role in Chapter 3 when we construct classical solutions of (**SIDE**). Specifically, in Chapter 2, we derive moment estimates and a strong limit theorem for space inverses of stochastic flows generated by jump SDEs with adapted coefficients in weighted Hölder norms. We also give a relatively simple, but novel, proof of existence and uniqueness of classical solutions of a degenerate SPDE (i.e. (**SIDE**) with $a^{1;ij} = \sigma^{i\varrho}\sigma^{j\varrho}$, $a^2 \equiv H^1 \equiv H^2 \equiv \rho^1 \equiv \rho^2 \equiv b \equiv c \equiv f \equiv \upsilon \equiv g \equiv h \equiv 0$, and $\varphi(x) = x$). Our method of proof allows us to prove existence and uniqueness when σ can degenerate on set of positive probability under less regularity than done previously. If σ is non-degenerate, then much more can be done, and we refer the reader to the seminal work [FGP10].

The focus of Chapter 3 is to give a complete proof of existence and uniqueness of classical solutions of (**SIDE**) with $a^{ij} = \sigma^{i\varrho}\sigma^{j\varrho} + \sigma^{2;i\varrho}\sigma^{2;j\varrho}$ and with Hölder assumptions on the coefficients and data using Feynman-Kac transformations, conditioning, and the interlacing of space-inverses of stochastic flows associated with the equations. We emphasize that σ, σ^2, H, and H^2 can vanish on a set of positive probability.

In Chapter 4, we prove the existence and uniqueness of solutions of degenerate linear stochastic evolution equations driven by jump processes in a Hilbert scale using the variational framework of stochastic evolution equations and the method of vanishing viscosity. This result extends the work of B. Rozovskii in [Roz90] from the continuous case to the jump case. As an application of our result, we derive unique solutions of (**SIDE**) with $a^{1;ij} \equiv a^{2;ij} \equiv \sigma \equiv g \equiv 0$, $E^2 = E^1 = V^1 = \emptyset$, and $\alpha \in (1,2)$ in the integer L^2-Sobolev scale. We confine ourself to an equation without continuous noise and diffusion because the theory of Sobolev solutions for equations in this case are well-studied (see, e.g. [KR82], [Roz90], and [GGK14]). The results derived in Chapters 1, 2, and 3 are the fruit of a collaborative effort with Remigijus Mikulevičius at the University of Southern California; both R. Mikulevičius and the author have contributed substantially to the development of the ideas contained in these chapters.

Finally, in Chapter 5, we consider finite difference schemes for (**SIDE**) with $d_2 = 1$, $H^1_t(x,z) = z$, $\rho^1 = 1$, $E^1 = V^1 = \emptyset$, $\alpha = 2$, $h \equiv H^2 \equiv \rho^2 \equiv 0$, and $a^{ij} - \sigma^{i\varrho}\sigma^{j\varrho} > 0$. We show that the $L^2(\Omega)$-pointwise rate of convergence is of order one in space and order one-half in time. The results given in this chapter are the fruit of a collaborative effort with Konstantinos Dareiotis at the University of Edinburgh; both K. Dareiotis and the author have contributed substantially to the development of the ideas contained in this chapter.

Basic Notation

Let $\mathbf{N}$ be the set of natural numbers and $\mathbf{N}_0 = \{0,1,\ldots\}$ be the set of natural numbers including zero. Let $\mathbf{Z}$ be the set of integers. For a topological space $(X,\mathcal{X})$, we denote the Borel sigma-field on X by $\mathcal{B}(X)$.

For each integer $n \geq 1$, let $\mathbf{R}^n$ be the n-dimensional Euclidean space and for each $x \in \mathbf{R}^n$, denote by $|x|$ the Euclidean norm of $x = (x_1,\ldots,x_n)$. We set $\mathbf{R} = \mathbf{R}^1$ and for two elements of $x, y \in \mathbf{R}$, we denote by $x \vee y$ the maximum of x and y and by $x \wedge y$, the minimum of x and y. Let $\mathbf{R}_+$ denote the set of non-negative elements of $\mathbf{R}^1$.

For an integer $n \geq 1$ and for each $i \in \{1,\ldots,n\}$, we let $\partial_i = \frac{\partial}{\partial x_i}$ be the spatial derivative operator with respect to the direction x_i and write $\partial_{ij} = \partial_i\partial_j$ for each $i, j \in \{1,\ldots,n\}$. We also denote the identity operator by ∂_0. For a once differentiable function $f = (f^1 \ldots, f^n) : \mathbf{R}^n \to \mathbf{R}^n$, we denote the gradient of f by $\nabla f = (\partial_j f^i)_{1\leq i,j\leq d_1}$. For a multi-index $\gamma = (\gamma_1,\ldots,\gamma_d) \in \{0,1,2,\ldots,\}^{d_1}$ of length $|\gamma| := \gamma_1 + \cdots + \gamma_d$, denote by ∂^γ the operator $\partial^\gamma = \partial_1^{\gamma_1}\cdots\partial_d^{\gamma_d}$, where ∂_i^0 is the identity operator for all $i \in \{1,\ldots,d_1\}$. For $i \in \{1,\ldots,d\}$, let $\partial_{-i} = -\partial_i$.

For integers $d_1, d_2 \geq 1$, we denote by $C_c^\infty(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$ the space of $\mathbf{R}^{d_2}$-valued infinitely differentiable functions with compact support in $\mathbf{R}^{d_1}$.

For each integer $n \geq 1$, the norm of an element x of $\ell_2(\mathbf{R}^n)$, the space of square-summable $\mathbf{R}^n$-valued sequences, is denoted by $|x|$. We set $\ell_2 = \ell_2(\mathbf{R})$. For integers $n \geq 1$

and a once differentiable function $f = (f^{1\varrho}, \ldots, f^{n\varrho})_{\varrho \in \mathbf{N}} : \mathbf{R}^n \to \ell_2(\mathbf{R}^n)$, we denote the gradient of f by $\nabla f = (\partial_j f^{i\varrho})_{1 \le i,j \le n, \varrho \in \mathbf{N}}$ and understand it as a function from $\mathbf{R}^{d_1}$ to $\ell_2(\mathbf{R}^{n^2})$.

For a Fréchet space χ and fixed time $T > 0$, we denote by $D([0,T];\chi)$ the space of χ-valued càdlàg (continuous from the right and limits from the left) functions on $[0,T]$ and by $C([0,T]^2;\chi)$ the space of χ-valued continuous functions on $[0,T] \times [0,T]$. The spaces $D([0,T];\chi)$ and $C([0,T]^2;\chi)$ are Fréchet spaces when endowed with the set of the supremum (in time) seminorms, which we assume, unless otherwise specified.

The notation $N = N(\cdot, \cdots, \cdot)$ is used to denote a positive constant depending only on the quantities appearing in the parentheses. In a given context, the same letter is often used to denote different constants depending on the same parameter. Moreover, for any function f defined on a set X and taking valued in a linear space Y with zero element 0_Y, the notation $f \equiv 0$ indicates that $f(x) = 0_Y$ for all $x \in X$.

Chapter 2

Properties of space inverses of stochastic flows

2.1 Introduction

Let $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ be a complete filtered probability space satisfying the usual conditions of right-continuity and completeness. Let w_t^ϱ, $t \geq 0$, $\varrho \in \mathbf{N}$, be a sequence of independent one-dimensional $\mathbf{F}$-adapted Wiener processes. For a sigma-finite measure space $(Z, \mathcal{Z}, \pi)$, we let $p(dt, dz)$ be an $\mathbf{F}$-adapted Poisson random measure on $(\mathbf{R}_+ \times Z, \mathcal{B}(\mathbf{R}_+) \otimes \mathcal{Z})$ with intensity measure $\pi(dz)dt$ and denote by $q(dt, dz) = p(dt, dz) - \pi(dz)dt$ the compensated Poisson random measure. For each real number $T > 0$, we let $\mathcal{R}_T$ and $\mathcal{P}_T$ be the $\mathbf{F}$-progressive and $\mathbf{F}$-predictable sigma-algebra on $\Omega \times [0, T]$, respectively.

Fix a real number $T > 0$ and an integer $d \geq 1$. For each stopping time $\tau \leq T$, consider the flow $X_t = X_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^d$, generated by the SDE

$$\begin{aligned} dX_t &= b_t(X_t)dt + \sigma_t^\varrho(X_t)dw_t^\varrho + \int_Z H_t(X_{t-}, z)q(dt, dz), \;\; \tau < t \leq T, \\ X_t &= x, \;\; t \leq \tau, \end{aligned} \tag{2.1.1}$$

where $b_t(x) = (b_t^i(\omega, x))_{1\leq i\leq d}$ and $\sigma_t(x) = (\sigma_t^{i\varrho}(\omega, x))_{1\leq i\leq d, \varrho\in\mathbf{N}}$ are $\mathbf{R}^d$-valued $\mathcal{R}_T \otimes \mathcal{B}(\mathbf{R}^d)$-measurable functions defined on $\Omega \times [0, T] \times \mathbf{R}^d$ and $H_t(x, z) = (H_t^i(\omega, x, z))_{1\leq i\leq d}$ is an $\mathbf{R}^d$-valued $\mathcal{P}_T \otimes \mathcal{B}(\mathbf{R}^d) \otimes \mathcal{Z}$-measurable function defined on $\Omega \times [0, T] \times \mathbf{R}^d \times Z$. The summation convention with respect to the repeated index ϱ is used here and below.

In this chapter, under Hölder regularity conditions on the coefficients b, σ, and H, we provide a simple and direct derivation of moment estimates of the space inverse of the flow, denoted $X_t^{-1}(\tau, x)$, in weighted Hölder norms. This is done by applying the Sobolev embedding theorem and the change of variable formula. Using a similar method, we establish a strong limit theorem in weighted Hölder norms for a sequence of flows $X_t^{(n)}(\tau, x)$ and their inverses $X_t^{(n);-1}(\tau, x)$ corresponding to a sequence of coefficients $(b^{(n)}, \sigma^{(n)}, H^{(n)})$ converging in weighted Hölder norms. Furthermore, as an application of the diffeomorphism property of flow, we give a direct derivation of the linear second order degenerate SPDE governing the inverse flow $X_t^{-1}(\tau, x)$ when $H \equiv 0$. Specifically, for each $\tau \leq T$, consider the stochastic flow $Y_t = Y_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^d$, generated by the SDE

$$
\begin{aligned}
dY_t &= b_t(Y_t)dt + \sigma_t^{\varrho}(Y_t)dw_t^{\varrho}, \quad \tau < t \le T, \\
Y_t &= x, \quad t \le \tau.
\end{aligned}
$$

Assume that b and σ have linear growth, bounded first and second derivatives, and that the second derivatives of b and σ are α-Hölder for some $\alpha > 0$. By partitioning the time interval and using Taylor's theorem, the Sobolev embedding theorem, and some basic properties of the flow and its inverse, we show that $u_t(x) = u_t(\tau, x) := Y_t^{-1}(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^d$ is the unique classical solution of the SPDE given by

$$
\begin{aligned}
du_t(x) &= \left(\frac{1}{2}\sigma_t^{i\varrho}(x)\sigma_t^{j\varrho}(x)\partial_{ij}u_t(x) - \hat{b}_t^i(x)\partial_i u_t(x)\right)dt - \sigma_t^{i\varrho}(x)\partial_i u_t(x)dw_t^{\varrho}, \quad \tau < t \le T, \\
u_t(x) &= x, \quad t \le \tau,
\end{aligned}
\tag{2.1.2}
$$

where

$$
\hat{b}_t^i(x) = b_t^i(x) - \sigma_t^{j\varrho}(x)\partial_j\sigma_t^{i\varrho}(x).
$$

In Chapter 3, we will use all of the properties of the flow $X_t(\tau, x)$ that are established in this first chapter in order to derive the existence and uniqueness of classical solutions of linear parabolic SIDEs.

One of the earliest works to investigate the homeomorphism property of flows of SDEs with jumps is by P. Meyer in [Mey81]. In [Mik83], R. Mikulevičius extended the properties found in [Mey81] to SDEs driven by arbitrary continuous martingales and random measures. Many other authors have since expanded upon the work in [Mey81], see for example [FK85, Kun04, MB07, QZ08, Zha13, Pri14] and references therein. In [Kun04, Kun86b], H. Kunita studied the diffeomorphism property of the flow $X_t(s, x)$, $(s, t, x) \in [0, T]^2 \times \mathbf{R}^d$, and in the setting of deterministic coefficients, he showed that for each fixed t, the inverse flow $X_t^{-1}(s, x)$, $(s, x) \in [t, T] \times \mathbf{R}^d$, solves a backward SDE. By estimating the associated backward SDE, one can obtain moment estimates and a strong limit theorem for the inverse flow in essentially the same way that moment estimates are obtained for the direct flow (see, e.g. [Kun86b]). However, this method of deriving moment estimates and a strong limit theorem for the inverse flow uses a time reversal, and thus ,requires that the coefficients are deterministic. In the case $H \equiv 0$, numerous authors have investigated properties of the inverse flow with random coefficients. In Chapter 2 of [Bis81], Lemma 2.1 and 2.2 of [OP89], and Section 6.1 and 6.2 of [Kun96], properties of $Y_t^{-1}(\tau, x)$ (i.e. moment estimates, strong limit theorem, and that it solves (2.1.2)) are established by first showing that the inverse flow solves the Stratonovich form SDE for

$Z_t = Z_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^d$, given by

$$dZ_t(x) = -U_t(Z_t(x))b_t(x)dt - U_t(Z_t(x))\sigma_t^{\varrho}(x) \circ dw_t^{\varrho}, \quad \tau < t \le T, \tag{2.1.3}$$
$$Z_0(x) = x, \quad \tau < t,$$

where $U_t(x) = U_t(\tau, x) = \nabla Y_t(\tau, x)^{-1}$. In order to obtain a strong solution of (2.1.3), conditions are imposed to ensure that $\nabla U_t(x)$ is locally-Lipschitz in x. In the degenerate setting, the third derivative of b_t and σ_t need to be α-Hölder for some $\alpha > 0$ to ensure that $\nabla U_t(x)$ is locally-Lipschitz in x. In direct contrast to this approach, we first derive properties of the inverse flow under the very assumptions that guarantee $Y_t(\tau, x)$ is a diffeomorphism (i.e. when the first derivative of the coefficients are α-Hölder for some $\alpha > 0$), and then we derive the existence of the equation without resorting to the SDE interpretation of the SPDE.

Classical solutions of (2.1.2) have been constructed in [Bis81, Kun96] by directly showing that $Y_t^{-1}(\tau, x)$ solves (2.1.3). As we have mentioned above, this approach requires the third derivatives of b_t and σ_t to be α-Hölder for some $\alpha > 0$. Yet another approach to deriving existence of classical solutions of (2.1.2) is using the method of time reversal (see, e.g. [Kun96, DPT98]). While this method only requires that the second derivatives of b_t and σ_t are α-Hölder for some $\alpha > 0$, it does impose that the coefficients are deterministic. In [KR82], N.V. Krylov and B.L. Rozvskii derived the existence and uniqueness of generalized solutions of degenerate second order linear parabolic SPDEs in Sobolev spaces using variational approach of SPDEs and the method of vanishing viscosity (see, also, [GGK14] and Chapter 4, Section 2, Theorem 1 in [Roz90]). Thus, by appealing to the Sobolev embedding theorem, this theory can be used to obtain classical solutions of degenerate linear SPDEs. Proposition 1 of Chapter 5, Section 2, in [Roz90] shows that if σ is uniformly bounded and four-times continuously differentiable in x with uniformly bounded derivatives and b is uniformly bounded and three-times continuously differentiable with uniformly bounded derivatives, then there exists a classical solution of (2.1.2) and $u_t(x) = Y_t^{-1}(x)$. This is more regularity than we require and we are also able to obtain solutions in the entire Hölder scale.

This chapter is organized as follows. In Section 2.2, we state our notation and main results. Section 2.3 is devoted to deriving moment estimates and a strong limit theorem for the space inverse of a stochastic flow generated by a Lévy driven SDE. In Section 2.4, we show that $Y_t^{-1}(\tau, x)$ is the unique classical solution of (2.1.2). In Section 2.5, the appendix, auxiliary facts that are used throughout the chapter are discussed.

2.2 Statement of main results

Let us describe some notation that will be used in this chapter. Elements of $\mathbf{R}^d$ are understood as column vectors and elements of $\mathbf{R}^{d^2}$ are understood as matrices of dimension $d \times d$. We denote the transpose of an element $x \in \mathbf{R}^d$ by x^*. For a Banach space V with norm $|\cdot|_V$, domain (i.e. open connected set) Q of $\mathbf{R}^d$, and continuous function $f : Q \to V$, we define

$$|f|_{0;Q;V} = \sup_{x \in Q} |f(x)|_V$$

and

$$[f]_{\beta;Q;V} = \sup_{x,y \in Q, x \neq y} \frac{|f(x) - f(y)|_V}{|x-y|_V^\beta}, \quad \beta \in (0,1].$$

For any real number $\beta \in \mathbf{R}$, we write $\beta = [\beta]^- + \{\beta\}^+$, where $[\beta]^-$ is an integer and $\{\beta\}^+ \in (0,1]$. For a Banach space V with norm $|\cdot|_V$, real number $\beta > 0$, and domain Q of $\mathbf{R}^d$, we denote by $C^\beta(Q;V)$ the Banach space of all bounded continuous functions $f : Q \to V$ having finite norm

$$|f|_{\beta;Q;V} := \sum_{|\gamma| \leq [\beta]^-} |\partial^\gamma f|_{0;Q;V} + \sum_{|\gamma| = [\beta]^-} [\partial^\gamma f]_{\{\beta\}^+;Q;V}.$$

When $Q = \mathbf{R}^d$ and $V = \mathbf{R}^n$ or $V = \ell_2(\mathbf{R}^n)$ for any integer $n \geq 1$, we drop the subscripts Q and V from the norm $|\cdot|_{\beta;Q;V}$ and write $|\cdot|_\beta$. For a Banach space V and for each $\beta > 0$, denote by $C^\beta_{\text{loc}}(\mathbf{R}^d;V)$ the Fréchet space of continuous functions $f : \mathbf{R}^d \to V$ satisfying $f \in C^\beta(Q;V)$ for all bounded domains $Q \subset \mathbf{R}^d$. We call a function $f : \mathbf{R}^d \to \mathbf{R}^d$ a $C^\beta_{\text{loc}}(\mathbf{R}^d;\mathbf{R}^d)$-diffeomorphism if f is a homeomorphism and both f and its inverse f^{-1} are in $C^\beta_{\text{loc}}(\mathbf{R}^d;\mathbf{R}^d)$.

If we do not specify to which space the parameters ω, t, x, y, z and n belong, then we mean $\omega \in \Omega$, $t \in [0,T]$, $x, y \in \mathbf{R}^d$, $z \in Z$, and $n \in \mathbf{N}$.

Let $r_1(x) = \sqrt{1 + |x|^2}$, $x \in \mathbf{R}^d$. For each real number $\beta > 1$, we introduce the following regularity condition on the coefficients b, σ, and H.

Assumption 2.2.1 (β)**.** *(1) There is a constant $N_0 > 0$ such that for all $(\omega,t,z) \in \Omega \times [0,T] \times Z$,*

$$|r_1^{-1} b_t|_0 + |\nabla b_t|_{\beta-1} + |r_1^{-1}\sigma_t|_0 + |\nabla \sigma_t|_{\beta-1} \leq N_0 \quad \textit{and} \quad |r_1^{-1} H_t(z)|_0 + |\nabla H_t(z)|_{\beta-1} \leq K_t(z),$$

where $K : \Omega \times [0,T] \times Z \to \mathbf{R}_+$ *is a* $\mathcal{P}_T \otimes \mathcal{Z}$*-measurable function satisfying*

$$K_t(z) + \int_Z K_t(z)^2 \pi(dz) \le N_0,$$

for all $(\omega, t, z) \in \Omega \times [0,T] \times Z$.

(2) There are constants $\eta \in (0,1)$ *and* $N_\kappa > 0$ *such that for all* $(\omega, t, x, z) \in \{(\omega, t, x, z) \in \Omega \times [0,T] \times \mathbf{R}^d \times Z : |\nabla H_t(\omega, x, z)| > \eta\}$,

$$|(I_d + \nabla H_t(x,z))^{-1}| \le N_\kappa.$$

The following theorem shows that if Assumption 2.2.1 (β) holds for some $\beta > 1$, then for any $\beta' \in [1, \beta]$, the solution $X_t(\tau, x)$ of (2.1.1) has a modification that is a $C^{\beta'}_{\mathrm{loc}}(\mathbf{R}^d\mathbf{R}^d)$ -diffeomorphism and the p-th moments of the weighted β'-Hölder norms of the inverse flow are bounded. This theorem will be proved in the next section.

Theorem 2.2.1. *Let Assumption 2.2.1*(β) *hold for some* $\beta > 1$.

(1) For any stopping time $\tau \le T$ *and* $\beta' \in [1, \beta)$*, there exists a modification of the strong solution* $X_t(\tau, x)$ *of* (2.1.1)*, also denoted by* $X_t(\tau, x)$*, such that* $\mathbf{P}$*-a.s. the mapping* $X_t(\tau, \cdot) : \mathbf{R}^d \to \mathbf{R}^d$ *is a* $C^{\beta'}_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{R}^d)$*-diffeomorphism,* $X_\cdot(\tau, \cdot), X^{-1}_\cdot(\tau, \cdot) \in D([0,T]; C^{\beta'}_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{R}^d))$*, and* $X^{-1}_{t-}(\tau, \cdot)$ *coincides with the inverse of* $X_{t-}(\tau, \cdot)$*. Moreover, for all* $\epsilon > 0$ *and* $p \ge 2$*, there is a constant* $N = N(d, p, N_0, T, \beta', \epsilon)$ *such that*

$$\mathbf{E}\left[\sup_{t \le T} |r_1^{-(1+\epsilon)} X_t(\tau)|_0^p\right] + \mathbf{E}\left[\sup_{t \le T} |r_1^{-\epsilon} \nabla X_t(\tau)|_{\beta'-1}^p\right] \le N$$

and a constant $N = N(d, p, N_0, T, \beta', \eta, N_\kappa, \epsilon)$ *such that*

$$\mathbf{E}\left[\sup_{t \le T} |r_1^{-(1+\epsilon)} X_t^{-1}(\tau)|_0^p\right] + \mathbf{E}\left[\sup_{t \le T} |r_1^{-\epsilon} \nabla X_t^{-1}(\tau)|_{\beta'-1}^p\right] \le N. \tag{2.2.1}$$

(2) If $H \equiv 0$*, then for all* $\beta' \in (1, \beta)$*,* $\mathbf{P}$*-a.s.* $X_\cdot(\cdot, \cdot), X^{-1}_\cdot(\cdot, \cdot) \in C([0,T]^2; C^{\beta'}_{\mathrm{loc}}(\mathbf{R}^d; \mathbf{R}^d))$ *and for all* $\epsilon > 0$ *and* $p \ge 2$*, there is a constant* $N = N(d, p, N_0, T, \beta', \epsilon)$ *such that*

$$\mathbf{E}\left[\sup_{s,t \le T} |r_1^{-(1+\epsilon)} X_t(s)|_0^p\right] + \mathbf{E}\left[\sup_{s,t \le T} |r_1^{-\epsilon} \nabla X_t(s)|_{\beta'-1}^p\right] \le N$$

and

$$\mathbf{E}\left[\sup_{s,t \le T} |r_1^{-(1+\epsilon)} X_t^{-1}(s)|_0^p\right] + \mathbf{E}\left[\sup_{s,t \le T} |r_1^{-\epsilon} \nabla X_t^{-1}(s)|_{\beta'-1}^p\right] \le N.$$

Remark 2.2.2. The estimate (2.2.1) is used in Chapter 3 to take the optional projection of a linear transformation of the inverse flow of a jump SDE driven by two independent

Weiner processes and two independent Poisson random measures relative to the filtration generated by one of the Weiner processes and Poisson random measures.

Now, let us state our strong limit theorem for a sequence of flows, which will also be proved in the next section. We will use this strong limit theorem in [LM14a] to show that the inverse flow of a jump SDE solves a parabolic SIDE. For each n, consider the stochastic flow $X_t^{(n)} = X_t^{(n)}(\tau, x)$, $(t,x) \in [0,T] \times \mathbf{R}^d$, generated by the SDE

$$
\begin{aligned}
dX_t^{(n)} &= b_t^{(n)}(X_t^{(n)})dt + \sigma_t^{(n)l\varrho}(X_t^{(n)})dw_t^{\varrho} + \int_Z H_t^{(n)}(X_{t-}^{(n)}, z)q(dt,dz), \quad \tau \le t \le T,\\
X_t^{(n)} &= x, \quad t \le \tau.
\end{aligned}
$$

Here we assume that for all n, $b^{(n)}$, $\sigma^{(n)}$, and $H^{(n)}$ satisfy the same measurability conditions as b, σ, and H, respectively.

Theorem 2.2.3. *Let Assumption 2.2.1(β) hold for some $\beta > 1$ and assume that $b^{(n)}, \sigma^{(n)}$, and $H^{(n)}$ satisfy Assumption 2.2.1 (β) uniformly in $n \in \mathbf{N}$. Moreover, assume that*

$$
d\mathbf{P}dt - \lim_{n\to\infty} \left(|r_1^{-1} b_t^{(n)} - r_1^{-1} b_t|_0 + |\nabla b_t^{(n)} - \nabla b_t|_{\beta-1} \right) = 0,
$$

$$
d\mathbf{P}dt - \lim_{n\to\infty} \left(|r_1^{-1} \sigma_t^{(n)} - r_1^{-1} \sigma_t|_{\beta-1} + |\nabla \sigma_t^{(n)} - \nabla \sigma_t|_0 \right) = 0,
$$

and for all $(\omega, t, z) \in \Omega \times [0,T] \times Z$ and $n \in \mathbf{N}$,

$$
|r_1^{-1} H_t^{(n)}(z) - r_1^{-1} H_t(z)|_0 + |\nabla H_t^{(n)}(z) - \nabla H_t(z)|_{\beta-1} \le K^{(n)}(t,z),
$$

where $(K_t^{(n)}(z))_{n\in\mathbf{N}}$ is a sequence of $\mathbf{R}_+$-valued $\mathcal{P}_T \otimes \mathcal{Z}$ measurable functions defined on $\Omega \times [0,T] \times Z$ satisfying for all $(\omega, t, z) \in \Omega \times [0,T] \times Z$ and $n \in \mathbf{N}$,

$$
K_t^{(n)}(z) + \int_Z K_t^{(n)}(z)^2 \pi(dz) \le N_0
$$

and

$$
d\mathbf{P}dt - \lim_{n\to\infty} \int_Z K_t^{(n)}(z)^2 \pi(dz) = 0.
$$

Then for any stopping time $\tau \le T$, $\beta' \in [1,\beta)$, and all $\epsilon > 0$, and $p \ge 2$, we have

$$
\lim_{n\to\infty} \left(\mathbf{E}\left[\sup_{t\le T} |r_1^{-(1+\epsilon)} X_t^{(n)}(\tau) - r_1^{-(1+\epsilon)} X_t(\tau)|_0^p \right] + \mathbf{E}\left[\sup_{t\le T} |r_1^{-\epsilon} \nabla X_t^{(n)}(\tau) - r_1^{-\epsilon} \nabla X_t(\tau)|_{\beta'-1}^p \right] \right) = 0,
$$

$$
\lim_{n\to\infty} \mathbf{E}\left[\sup_{t\le T} |r_1^{-(1+\epsilon)} X_t^{(n);-1}(\tau) - r_1^{-(1+\epsilon)} X_t^{-1}(\tau)|_0^p \right] = 0,
$$

and

$$
\lim_{n\to\infty} \mathbf{E}\left[\sup_{t\le T} |r_1^{-\epsilon} \nabla X_t^{(n);-1}(\tau) - r_1^{-\epsilon} \nabla X_t^{-1}(\tau)|_{\beta'-1}^p \right] = 0.
$$

Let us introduce our class of solutions for the equation (2.1.1). Let O_T be the $\mathbf{F}$-optional sigma-algebra on $\Omega \times [0,T]$. For a each number $\beta' > 2$, let $\mathfrak{C}^{\beta'}_{\text{cts}}(\mathbf{R}^d;\mathbf{R}^d)$ be the linear space of all random fields $v : \Omega \times [0,T] \times \mathbf{R}^d \to \mathbf{R}^d$ such that v is $O_T \otimes \mathcal{B}(\mathbf{R}^d)$-measurable and $\mathbf{P}$-a.s. $r_1^{-\lambda}(\cdot)v_\cdot(\cdot)$ is a $C([0,T];C^{\beta'}(\mathbf{R}^d;\mathbf{R}^d))$ for a real number $\lambda > 0$.

We introduce the following assumption for a real number $\beta > 2$.

Assumption 2.2.2 (β)**.** *There is a constant N_0 such that for all $(\omega,t) \in \Omega \times [0,T]$,*

$$|r_1^{-1}b_t|_0 + |r_1^{-1}\sigma_t|_0 + |\nabla b_t|_{\beta-1} + |\nabla \sigma_t|_{\beta-1} \le N_0.$$

Theorem 2.2.4. *Let Assumption 2.2.2(β) hold for some $\beta > 2$. Then for any stopping time $\tau \le T$ and $\beta' \in [1,\beta)$, there exists a unique process $u(\tau)$ in $\mathfrak{C}^{\beta'}_{\text{cts}}(\mathbf{R}^d;\mathbf{R}^d)$ that solves (2.1.2). Moreover, $\mathbf{P}$-a.s. $u_t(\tau,x) = Y_t^{-1}(\tau,x)$ for all $(t,x) \in [0,T] \times \mathbf{R}^d$ and for all $\epsilon > 0$ and $p \ge 2$, there is a constant $N = N(d,p,N_0,T,\beta',\epsilon)$ such that*

$$\mathbf{E}\left[\sup_{s,t\le T}|r_1^{-(1+\epsilon)}u_t(s)|_0^p\right] + \mathbf{E}\left[\sup_{s,t\le T}|r_1^{-\epsilon}\nabla u_t(s)|_{\beta'-1}^p\right] \le N.$$

Remark 2.2.5. It is clear by the proof of this theorem that if $\sigma \equiv 0$, then we only need to assume that Assumption 2.2.2 (β) holds for some $\beta > 1$.

Now let us consider the SPDE given by

$$\begin{aligned} d\bar{u}_t(x) &= \left(\frac{1}{2}\sigma_t^{i\varrho}(x)\sigma_t^{j\varrho}(x)\partial_{ij}\bar{u}_t(x) + b_t^i(x)\partial_i\bar{u}_t(x)\right)dt + \sigma_t^{i\varrho}(x)\partial_i\bar{u}_t(x)dw_t^\varrho, \quad \tau < t \le T, \\ \bar{u}_t(x) &= x, \quad t \le \tau. \end{aligned} \tag{2.2.2}$$

This SPDE differs from the one given in (2.1.2) by the first-order coefficient in the drift. In order to obtain an existence and uniqueness theorem for this equation, we have to impose additional assumptions on σ.

We introduce the following assumption for a real number $\beta > 2$.

Assumption 2.2.3 (β)**.** *There is a constant $N_0 > 0$ such that for all $(\omega,t) \in \Omega \times [0,T]$,*

$$|r_1^{-1}b_t|_0 + |\nabla b_t|_{\beta-1} + |\sigma_t|_{\beta+1} \le N_0.$$

For each $\tau \le T$, consider the stochastic flow $\hat{Y}_t = \hat{Y}_t(\tau,x)$, $(t,x) \in [0,T]\times\mathbf{R}^d$, generated by the SDE

$$\begin{aligned} d\bar{Y}_t &= -\hat{b}_t(\bar{Y}_t)dt - \sigma_t^\varrho(\bar{Y}_t)dw_t^\varrho, \quad \tau < t \le T, \\ Y_t &= x, \quad t \le \tau. \end{aligned}$$

If Assumption 2.2.3(β) holds for some $\beta > 2$, then for all $(\omega, t, x) \in \Omega \times [0, T] \times \mathbf{R}^d$,

$$|\hat{b}_t(x)| \leq |b_t(x)| + |\sigma_t(x)| \nabla\sigma_t(x)| \leq N_0(N_0 + 1) + N_0|x|$$

and

$$|\nabla\hat{b}_t|_{\beta-1} \leq |\nabla b_t|_{\beta-1} + |\sigma_t|_{\beta-1}|\nabla^2\sigma_t|_{\beta-1} + |\nabla\sigma_t|^2_{\beta-1} \leq N_0 + 2N_0^2,$$

which immediately implies the following corollary of Theorem 2.2.4.

Corollary 2.2.6. *If Assumption 2.2.3(β) holds for some $\beta > 2$, then for any stopping time $\tau \leq T$ and $\beta' \in [1, \beta)$, there exists a unique process $\bar{u}(\tau)$ in $\mathfrak{C}^{\beta'}_{\mathrm{cts}}(\mathbf{R}^d; \mathbf{R}^d)$ that solves (2.2.2). Moreover, $\mathbf{P}$-a.s. $\bar{u}_t(\tau, x) = \bar{Y}_t^{-1}(\tau, x)$ for all $(t, x) \in [0, T] \times \mathbf{R}^d$ and for all $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d, p, N_0, T, \beta', \epsilon)$ such that*

$$\mathbf{E}\left[\sup_{s,t \leq T} |r_1^{-(1+\epsilon)}\bar{u}_t(s)|_0^p\right] + \mathbf{E}\left[\sup_{s,t \leq T} |r_1^{-\epsilon}\nabla\bar{u}_t(s)|^p_{\beta'-1}\right] \leq N.$$

2.3 Properties of stochastic flows

2.3.1 Homeomorphism property of flows

In this subsection, we collect some results about flows of jump SDEs that we will need. In particular, we present sufficient conditions that guarantee the homeomorphism property of flows of jump SDEs. First, let us introduce the following assumption, which is the usual linear growth and Lipschitz condition on the coefficients b, σ, and H of the SDE (2.1.1).

Assumption 2.3.1. *There is a constant $N_0 > 0$ such that for all $(\omega, t, x, y) \in \Omega \times [0, T] \times \mathbf{R}^{2d}$,*

$$|b_t(x)| + |\sigma_t(x)| \leq N_0(1 + |x|),$$
$$|b_t(x) - b_t(y)| + |\sigma_t(x) - \sigma_t(y)| \leq N_0|x - y|.$$

Moreover, for all $(\omega, t, x, y, z) \in \Omega \times [0, T] \times \mathbf{R}^{2d} \times Z$,

$$|H_t(x, z)| \leq K_1(t, z)(1 + |x|),$$
$$|H_t(x, z) - H_t(y, z)| \leq K_2(t, z)|x - y|,$$

where $K_1, K_2 : \Omega \times [0, T] \times Z \to \mathbf{R}_+$ are $\mathcal{P}_T \otimes \mathcal{Z}$-measurable functions satisfying

$$K_1(t, z) + K_2(t, z) + \int_Z \left(K_1(t, z)^2 + K_2(t, z)^2\right)\pi(dz) \leq N_0,$$

for all $(\omega, t, z) \in \Omega \times [0, T] \times Z$.

It is well-known that under this assumption, there exists a unique strong solution $X_t(s,x)$ of (2.1.1) (see e.g. Theorem 3.1 in [Kun04]). We will also make use of the following assumption.

Assumption 2.3.2. *For all* $(\omega,t,x,z)\in\Omega\times[0,T]\times\mathbf{R}^d\times Z$, $H_t(x,z)$ *is differentiable in* x, *and there are constants* $\eta\in(0,1)$ *and* $N_\kappa>0$ *such that for all* $(\omega,t,x,z)\in\{(\omega,t,x,z)\in\Omega\times[0,T]\times\mathbf{R}^d\times Z : |\nabla H_t\,(\omega,x,z)|>\eta\}$,

$$\left|(I_d+\nabla H_t(x,z))^{-1}\right|\leq N_\kappa.$$

The coming lemma shows that under Assumptions 2.3.1 and 2.3.2, the mapping $x+H_t(x,z)$ from $\mathbf{R}^d$ to $\mathbf{R}^d$ is a diffeomorphism and the gradient of inverse map is bounded.

Lemma 2.3.1. *Let Assumptions 2.3.1 and 2.3.2 hold. For all* $(\omega,t,z)\in\Omega\times[0,T]\times Z$, *the mapping* $\tilde{H}_t(\cdot,z):\mathbf{R}^d\to\mathbf{R}^d$ *defined by* $\tilde{H}_t(x,z):=x+H_t(x,z)$ *is a diffeomorphism and*

$$|\tilde{H}_t^{-1}(x,z)|\leq\bar{N}N_0+\bar{N}|x|\quad\text{and}\quad|\nabla\tilde{H}_t^{-1}(x,z)|\leq\bar{N},$$

where $\bar{N}:=(1-\eta)^{-1}\vee N_0$.

Proof. (1) On the set $(\omega,t,x,z)\in\{(\omega,t,x,z)\in\Omega\times[0,T]\times\mathbf{R}^d\times Z : |\nabla H_t(\omega,x,z)|\leq\eta\}$, we have

$$|\kappa_t(\omega,x,z)|\leq\left|I_d+\sum_{n=1}^{\infty}(-1)^n[\nabla H_t(\omega,x,z)]^n\right|\leq\frac{1}{1-\eta}.$$

It follows from Assumption 2.3.2 that for all ω,t,x, and z, the mapping $\nabla\tilde{H}_t(x,z)$ has a bounded inverse. Therefore, by Theorem 0.2 in [DMGZ94], the mapping $\tilde{H}_t(\cdot,z):\mathbf{R}^d\to\mathbf{R}^d$ is a global diffeomorphism. Moreover, for all ω,t,x and z,

$$|\tilde{H}_t^{-1}(x,z)-\tilde{H}_t^{-1}(y,z)|\leq\bar{N}|x-y|,$$

which yields

$$|\tilde{H}_t(x,z)-\tilde{H}_t(y,z)|\geq\bar{N}^{-1}|x-y|\quad\Longrightarrow\quad|\tilde{H}_t(x,z)|+K_1(t,z)\geq\bar{N}^{-1}|x|,$$

and hence

$$|\tilde{H}_t^{-1}(x,z)|\leq\bar{N}K_1(t,z)+\bar{N}|x|\leq\bar{N}N_0+\bar{N}|x|.$$

□

The following estimates are essential in the proof of the homeomorphic property of the flow and the derivation of moment estimates of the inverse flow. We refer the reader to Theorem 3.2 and Lemmas 3.7 and 3.9 in [Kun04] and Lemma 4.5.6 in [Kun97] ($H\equiv 0$ case) for the proof of the following lemma.

Lemma 2.3.2. *Let Assumption 2.3.1 hold.*

(1) For all $p \geq 2$, there is a constant $N = N(p, N_0, T)$ such that for all $s, \bar{s} \in [0,T]$ and $x, y \in \mathbf{R}^d$,

$$\mathbf{E}\left[\sup_{t\leq T} r_1(X_t(s,x))^p\right] \leq N r_1(x)^p, \tag{2.3.1}$$

$$\mathbf{E}\left[\sup_{t\leq T} |X_t(s,x) - X_t(s,y)|^p\right] \leq N|x-y|^p. \tag{2.3.2}$$

(2) If Assumption 2.3.2 holds, then for any $p \in \mathbf{R}$, there is a constant $N = N(p, N_0, T, \eta, N_\kappa)$ such that for all $s \in [0,T]$ and $x, y \in \mathbf{R}^d$,

$$\mathbf{E}\left[\sup_{t\leq T} r_1(X_t(s,x))^p\right] \leq N r_1(x)^p, \tag{2.3.3}$$

and

$$\mathbf{E}\left[\sup_{t\leq T} |X_t(s,x) - X_t(s,Y)|^p\right] \leq N|x-y|^p. \tag{2.3.4}$$

In the next proposition, we collect some facts about the homeomorphic property of the flow. Let us mention that the homeomorphism property has been shown in [QZ08] to hold under a log-Lipschitz condition on the coefficients. The key idea is to use Bihari's inequality instead of Gronwall's inequality, but we do not pursue this here.

Proposition 2.3.3. *Let Assumptions 2.3.1 and 2.3.2 hold.*

(1) There exists a modification of the strong solution $X_t(s,x)$, $(s,t,x) \in [0,T]^2 \times \mathbf{R}^d$, of (2.1.1), also denoted by $X_t(s,x)$, that is càdlàg in s and t and continuous in x. Moreover, for any stopping time $\tau \leq T$, $\mathbf{P}$-a.s. for all $t \in [0,T]$, the mappings $X_t(\tau,\cdot), X_{t-}(\tau,\cdot) : \mathbf{R}^d \to \mathbf{R}^d$ are homeomorphisms and the inverse of $X_t(\tau,\cdot)$, denoted by $X_t^{-1}(\tau,\cdot)$, is càdlàg in t and continuous in x, and $X_{t-}^{-1}(\tau,\cdot)$ coincides with the inverse of $X_{t-}(\tau,\cdot)$. In particular, if $(x_n)_{n\geq 1}$ is a sequence in $\mathbf{R}^d$ such that $\lim_{n\to\infty} x_n = x$ for some $x \in \mathbf{R}^d$, then $\mathbf{P}$-a.s.

$$\lim_{n\to\infty} \sup_{t\leq T} |X_t^{-1}(\tau, x_n) - X_t^{-1}(\tau, x)| = 0.$$

Furthermore, for all $\beta' \in [0,1)$, $\mathbf{P}$-a.s. $X(\tau,\cdot) \in D([0,T]; C^{\beta'}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d))$ and for all $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d, p, N_0, T, \beta', \epsilon)$ such that

$$\mathbf{E}\left[\sup_{t\leq T} |r_1^{-(1+\epsilon)} X_t(\tau)|^p_{\beta'}\right] \leq N. \tag{2.3.5}$$

(2) If $H \equiv 0$, then $\mathbf{P}$-a.s. for all $s, t \in [0,T]$, the $X_t(s,x)$ and $X_t^{-1}(s,x)$ are continuous in s, t, and x. Moreover, for all $\beta' \in [0,1]$, $\mathbf{P}$-a.s. $X(\cdot,\cdot) \in C([0,T]^2; C^{\beta'}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d))$ and for

all $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d, p, N_0, T, \beta', \epsilon)$ such that

$$\mathbf{E}\left[\sup_{s,t\leq T} |r_1^{-(1+\epsilon)} X_t(s)|_{\beta'}^p\right] \leq N. \tag{2.3.6}$$

Proof. (1) Owing to Assumptions 2.3.1 and 2.3.2, by Lemma 2.3.1, for all ω, t and z, the process $\tilde{H}_t(x,z) := x + H_t(x,z)$ is a homeomorphism (in fact, it is a diffeomorphism) in x and $\tilde{H}_t^{-1}(x,z)$ has linear growth and is Lipschitz. This implies that assumptions of Theorem 3.5 in [Kun04] hold and hence there is modification of $X_t(s,x)$, denoted $X_t(s,x)$, such that for all $s \in [0,T]$, $\mathbf{P}$-a.s. for all $t \in [0,T]$, $X_t(s,\cdot)$ is a homeomorphism. Following [Kun04], for each $(s,t,x) \in [0,T]^2 \times \mathbf{R}^d$, we set

$$\bar{X}_t(s,x) = \begin{cases} x & t \leq s \\ X_t(0, X_s^{-1}(0,x)) & t \geq s, \end{cases} \tag{2.3.7}$$

and remark that $\mathbf{P}$-a.s. $\bar{X}_t(s,x)$ is càdlàg in s and t and continuous in x, and $\mathbf{P}$-a.s. for all $(s,t) \in [0,T]^2$, $\bar{X}_t(s,\cdot)$ is a homeomorphism, and $\bar{X}_t(s,x)$ is a version of $X_t(s,x)$ (the equation started at s). Fix a stopping time $\tau \leq T$. We will now show that $\bar{X}_t(\tau,x) = \bar{X}_t(s,x)|_{s=\tau}$ (i.e. $\bar{X}_t(s,x)$ evaluated at $s=\tau$) is a version of $X_t(\tau,x)$. Define the sequence of stopping times $(\tau_n)_{n\geq 1}$ by

$$\tau_n = \sum_{k=1}^{n-1} \frac{kT}{n} \mathbf{1}_{\left\{\frac{(k-1)T}{n} \leq \tau < \frac{kT}{n}\right\}} + T\mathbf{1}_{\left\{\tau \geq \frac{(n-1)T}{n}\right\}}.$$

For each n and x, let $X_t^{(n)} = X_t^{(n)}(x) = \bar{X}_t(\tau_n, x)$, $t \in [0,T]$. It follows that for all n, t, and x, $\mathbf{P}$-a.s. for all $k \in \{1,\dots,n\}$,

$$X_t^{(n)}(x)\mathbf{1}_{\{\tau_n = \frac{kT}{n}\}} = X_t\left(\frac{kT}{n}, x\right)\mathbf{1}_{\{\tau_n = \frac{kT}{n}\}},$$

and hence

$$\begin{aligned} X_t^{(n)}(x)\mathbf{1}_{\{\tau_n=\frac{kT}{n}\}} &= \mathbf{1}_{\{\tau_n=\frac{kT}{n}\}}x + \mathbf{1}_{\{\tau_n=\frac{kT}{n}\}}\int_{]\frac{kT}{n},\frac{kT}{n}\vee t]} b_r(X_r^{(n)}(x))dr \\ &+ \mathbf{1}_{\{\tau_n=\frac{kT}{n}\}}\int_{]\frac{kT}{n},\frac{kT}{n}\vee t]} \sigma_r^{\varrho}(X_r^{(n)}(x))dw_r^{\varrho} \\ &+ \mathbf{1}_{\{\tau_n=\frac{kT}{n}\}}\int_{]\frac{kT}{n},\frac{kT}{n}\vee t]}\int_Z H_r(X_r^{(n)}(x),z)q(dr,dz). \end{aligned}$$

Since Ω is the disjoint union of the sets $\left\{\tau_n = \frac{kT}{n}\right\}$, $k \in \{1,\dots,n\}$, it follows that $X_t^{(n)}(x)$

solves

$$X_t^{(n)}(x) = x + \int_{]\tau_n,\tau_n\vee t]} b_r(X_r^{(n)}(x))dr + \int_{]\tau_n,\tau_n\vee t]} \sigma_r^{\varrho}(X_r^{(n)}(x))dw_r^{\varrho} \\ + \int_{]\tau_n,\tau_n\vee t]}\int_Z H_r(X_r^{(n)}(x),z)q(dr,dz).$$

Thus, by uniqueness, we find that for all t and x, $\mathbf{P}$-a.s. $\bar{X}_t(\tau_n,x) = X_t^{(n)}(x) = X_t(\tau_n,x)$. It is easy to check that for all t and x, $\mathbf{P}$-a.s. $X_t(\tau_n,x)$ converges to $X_t(\tau,x)$ as n tends to infinity. Since $\bar{X}_t(s,x)$ is càdlàg in s, $\bar{X}_t(\tau_n,x)$ converges to $\bar{X}_t(\tau,x)$ as n tends to infinity. Therefore, $\bar{X}_t(\tau,x)$ is a version of $X_t(\tau,x)$ for all t and x. We identify $X_t(s,x)$ and $\bar{X}_t(s,x)$ for all $(s,t,x) \in [0,T]^2 \times \mathbf{R}^d$. Using Lemma 2.3.2(1) and Corollary 2.5.3, we obtain that $\mathbf{P}$-a.s $X_\cdot(\tau,\cdot) \in D([0,T];C^{\beta'}_{\text{loc}}(\mathbf{R}^d;\mathbf{R}^d))$ and that the estimate (2.3.5) holds. Note here that for all $\beta \geq 0$, the Fréchet spaces $D([0,T];C^{\beta}_{\text{loc}}(\mathbf{R}^d;\mathbf{R}^d))$ and $C^{\beta}_{\text{loc}}(\mathbf{R}^d;D([0,T];\mathbf{R}^d))$ are equivalent. It follows from the proof of Theorem 3.5 in [Kun04] that for every stopping time $\bar{\tau} \leq T$, $\mathbf{P}$-a.s.

$$\lim_{|x|\to\infty} \inf_{t\leq T} |X_t(\bar{\tau},x)| = \infty. \tag{2.3.8}$$

Let $(t_n) \subseteq [0,T]$ and $(x_n) \subseteq \mathbf{R}^d$ be convergent sequences with limits t and x, respectively. First, assume $t_n < t$ for all n. By (2.3.8), for every stopping time $\bar{\tau} \leq T$, $\mathbf{P}$-a.s. the sequence $\left(X_{t_n}^{-1}(\bar{\tau},x_n)\right)$ is uniformly bounded. Since $\mathbf{P}$-a.s. $X_\cdot(\tau,\cdot) \in D([0,T];C^{\beta}(\mathbf{R}^d;\mathbf{R}^d))$, $\beta' \in (0,1)$, we have

$$\begin{aligned}\lim_{n\to\infty} \left(X_{t-}(\bar{\tau},X_{t_n}^{-1}(\bar{\tau},x_n)) - X_{t-}(\bar{\tau},X_{t-}^{-1}(\bar{\tau},x)\right) &= \lim_{n\to\infty} \left(X_{t-}(\bar{\tau},X_{t_n}^{-1}(\bar{\tau},x_n)) - x\right) \\ = \lim_{n\to\infty} \left(X_{t_n}(\bar{\tau},X_{t_n}^{-1}(\bar{\tau},x_n)) - x\right) &= \lim_{n\to\infty} (x_n - x) = 0,\end{aligned}$$

which implies

$$\lim_{n\to\infty} X_{t_n}^{-1}(\bar{\tau},x_n) = X_{t-}^{-1}(\bar{\tau},x).$$

A similar argument is used for $t_n > t$. (2) It follows from the definition (2.3.7) that $\bar{X}_t(s,x)$ and $\bar{X}_t^{-1}(s,x)$ are continuous in s, t, and x. Moreover, applying Lemma 2.3.2(1) and Corollary 2.5.3, we conclude that $\mathbf{P}$-a.s. $X_\cdot(\cdot,\cdot) \in C([0,T]^2;C^{\beta'}_{\text{loc}}(\mathbf{R}^d;\mathbf{R}^d))$ and that the estimate (2.3.6) holds. The continuity of $X_s(\tau,x)$ with respect to s plays an important role in the proof of Theorem 2.2.4. □

2.3.2 Moment estimates of inverse flows: Proof of Theorem 2.2.1

In this subsection, under Assumption 2.2.1(β), $\beta \geq 1$, we derive moment estimates for the flow $X_t(\tau,x)$ and its inverse $X_t^{-1}(\tau,x)$ in weighted Hölder norms and complete the proof of Theorem 2.2.1. In particular, we will apply Corollaries 2.5.2 and 2.5.3 with the Banach spaces $V = D([0,T];\mathbf{R}^d)$ and $V = C([0,T]^2;\mathbf{R}^d)$.

Proposition 2.3.4. *Let Assumption 2.2.1(β) hold for some $\beta > 1$*

(1) For all stopping time s$\tau \le T$ and $\beta' \in [1,\beta)$, $\mathbf{P}$-a.s. $\nabla X_{\cdot}(\tau,\cdot) \in D([0,T];C^{\beta'-1}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d))$ and for all $\epsilon > 0$ and $p \ge 2$, there is a constant $N = N(d,p,N_0,T,\beta',\epsilon)$ such that

$$\mathbf{E}\left[\sup_{t\le T}|r_1^{-\epsilon}\nabla X_t(\tau)|^p_{\beta'-1}\right] \le N. \tag{2.3.9}$$

Moreover, for all $p \ge 2$, there is a constant $N = N(d,p,N_0,\beta,T)$ such that for all multi-indices γ with $1 \le |\gamma| \le [\beta]$ and all $x \in \mathbf{R}^d$,

$$\mathbf{E}\left[\sup_{t\le T}|\partial^\gamma X_t(\tau,x)|^p\right] \le N \tag{2.3.10}$$

and for all multi-indices γ with $|\gamma| = [\beta]^-$ and all $x,y \in \mathbf{R}^d$,

$$\mathbf{E}\left[\sup_{t\le T}|\partial^\gamma X_t(\tau,x) - \partial^\gamma X_t(\tau,y)|^p\right] \le N|x-y|^{\{\beta\}^+p}. \tag{2.3.11}$$

(2) If $H \equiv 0$, then for all $\beta' \in [1,\beta)$, $\mathbf{P}$-a.s. $\nabla X_{\cdot}(\cdot,\cdot) \in C([0,T]^2;C^{\beta'-1}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d))$ and for all $\epsilon > 0$ and $p \ge 2$, there is a constant $N = N(d,p,N_0,T,\beta',\epsilon)$ such that

$$\mathbf{E}\left[\sup_{s,t\le T}|r_1^{-(1+\epsilon)}\nabla X_t(s)|^p_{\beta'-1}\right] \le N.$$

Moreover, for all $p \ge 2$, there is a constant $N = N(d,p,N_0,T,\beta)$ such that for all multi-indices γ with $|\gamma| = [\beta]^-$ and all $s,\bar{s} \in [0,T]$ and $x \in \mathbf{R}^d$,

$$\mathbf{E}\left[\sup_{t\le T}|\partial^\gamma X_t(s,x) - \partial^\gamma X_t(\bar{s},x)|^p\right] \le N|s-\bar{s}|^{p/2}. \tag{2.3.12}$$

Proof. (1) Fix a stopping time $\tau \le T$ and write $X_t(\tau,x) = X_t(x)$. First, let us assume that $[\beta]^- = 1$. It follows from Theorem 3.4 in [Kun04] that $\mathbf{P}$-a.s. for all t, $X_t(\tau,\cdot)$ is continuously differentiable and $U_t = \nabla X_t(\tau,x)$ satisfies

$$\begin{aligned}
&dU_t = \nabla b_t(X_t)U_t dt + \nabla\sigma^\varrho_t(X_{t-})U_t dw^\varrho_t + \int_Z \nabla H_t(X_{t-},z)U_{t-}q(dt,dz), \quad \tau < t \le T,\\
&\nabla X_t = I_d, \quad t \le \tau,
\end{aligned} \tag{2.3.13}$$

where I_d is the $d \times d$-dimensional identity matrix. Taking $\lambda = 0$ in the estimates (3.10) and (3.11) in Theorem 3.3 in [Kun04], we obtain (2.3.10) and (2.3.11). Then applying Corollary 2.5.3 with $V = D([0,T];\mathbf{R}^d)$, we have that $X_{\cdot}(\cdot) \in D([0,T];C^{\beta'}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d))$ and that (2.3.9) holds. The proof for $[\beta]^- > 1$ follows by induction (see, e.g. the proof of Theorem 6.4 in [Kun97]).

(2) The estimate (2.3.12) is given in Theorem 4.6.4 in [Kun97] in equation (19). The remaining items of part (2) then follow in exactly the same way as part (1) with the only exception being that we apply Corollary 2.5.3 with $V = C([0,T]^2;\mathbf{R}^d)$. □

Lemma 2.3.5. *Let Assumption 2.2.1(β) hold for some $\beta > 1$.*

(1) For any stopping time $\tau \le T$ and $\beta' \in [1,\beta)$, $\mathbf{P}$-a.s. $\nabla X_\cdot(\tau,\cdot)^{-1} \in D([0,T];C^{\beta'-1}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d))$ and for all $p \ge 2$, there is a constant $N = N(d,p,N_0,T,\eta,N_\kappa)$ such that for all $x,y \in \mathbf{R}^d$

$$\mathbf{E}\left[\sup_{t\le T}|\nabla X_t(\tau,x)^{-1}|^p\right] \le N \tag{2.3.14}$$

and

$$\mathbf{E}\left[\sup_{t\le T}|\nabla X_t(\tau,x)^{-1} - \nabla X_t(\tau,y)^{-1}|^p\right] \le N|x-y|^{((\beta-1)\wedge 1)p}. \tag{2.3.15}$$

(2) If $H \equiv 0$, then for all $\beta' \in [1,\beta)$, $\mathbf{P}$-a.s. $\nabla X_\cdot(\cdot,\cdot)^{-1} \in C([0,T]^2;C^{\beta'-1}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d))$ and for all $p \ge 2$, there is a constant $N = N(d,p,N_0,T)$ such that for all $s,\bar{s} \in [0,T]$ and $x \in \mathbf{R}^d$,

$$\mathbf{E}\left[\sup_{t\le T}|\nabla X_t(s,x)^{-1} - \nabla X_t(s,x)^{-1}|^p\right] \le N|s-\bar{s}|^{p/2}.$$

Proof. (1) Let $\tau \le T$ be a fixed stopping time and write $X_t(\tau,x) = X_t(x)$. Using Itô's formula (see also Lemma 3.12 in [Kun04]), we deduce that $\bar{U}_t = \nabla X_t(x)^{-1}$ satisfies

$$\begin{aligned}
d\bar{U}_t &= \bar{U}_t\left(\nabla\sigma^\varrho_t(X_{t-})\nabla\sigma^\varrho_t(X_{t-}(\tau)) - \nabla b_t(X_t)\right)dt - \bar{U}_t\nabla\sigma^\varrho_t(X_t)dw^\varrho_t \\
&\quad - \int_Z \bar{U}_{t-}\nabla H_t(X_{t-},z)(I_d + \nabla H_t(X_{t-},z))^{-1}q(dt,dz) \\
&\quad + \int_Z \bar{U}_t\nabla H_t(X_{t-},z)^2(I_d + \nabla H_t(X_{t-},z))^{-1}\pi(dz)dt, \quad \tau < t \le T, \\
\bar{U}_t &= I_d, \quad t \le \tau.
\end{aligned} \tag{2.3.16}$$

Since matrix inversion is a smooth mapping, the coefficients of the linear equation (2.3.16) satisfy the same assumptions as the coefficients of the linear equation (2.3.13), and hence the derivation of the estimates (2.3.14) and (2.3.15) proceed in the same way as the analogous estimates for (2.3.13). To see that $\mathbf{P}$-a.s. $X_\cdot(\cdot)^{-1} \in D([0,T];C^{\beta'-1}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d))$, we only need to note that $\mathbf{P}$-a.s. $X_\cdot(\cdot) \in D([0,T];C^{\beta'-1}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d))$ and that matrix inversion is a smooth mapping. Part (2) follows with the obvious changes. □

As an immediate corollary, we obtain the diffeomorphism property of the flow $X_t(\tau,x)$ under Assumption 2.2.1(β), $\beta > 1$.

Corollary 2.3.6. *Let Assumption 2.2.1(β) hold.*

(1) For all stopping times $\tau \leq T$ and $\beta' \in [1,\beta)$ the mapping $X_t(\tau,\cdot) : \mathbf{R}^d \to \mathbf{R}^d$ is a $C^{\beta'}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d)$-diffeomorphism, $\mathbf{P}$-a.s. $X_\cdot(\tau,\cdot), X^{-1}_\cdot(\tau,\cdot) \in D([0,T];C^{\beta'}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d))$ and for all $t \in [0,T]$, $X^{-1}_{t-}(\tau)$ coincides with the inverse of $X_{t-}(\tau)$.
(2) If $H \equiv 0$, then for all $\beta' \in [1,\beta)$, $\mathbf{P}$-a.s. $X_\cdot(\cdot,\cdot), X^{-1}_\cdot(\cdot,\cdot) \in C([0,T]^2, C^{\beta'}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d))$.

Proof. (1) Fix a stopping time $\tau \leq T$ and write $X_t(\tau,x) = X_t(x)$. It follows from Propositions 2.3.3 and 2.3.4 that $\mathbf{P}$-a.s. for all t, the mappings $X_t(\cdot), X_{t-}(\cdot) : \mathbf{R}^d \to \mathbf{R}^d$ are homeomorphisms and $X_\cdot(\cdot) \in D([0,T];C^{\beta'}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d))$. Moreover, by Lemma 2.3.5, $\mathbf{P}$-a.s. for all t and x, the matrix $\nabla X_t(\tau,x)$ has an inverse. Therefore, by Hadamard's Theorem (see, e.g., Theorem 0.2 in [DMGZ94]), $\mathbf{P}$-a.s. for all t, $X_t(\cdot)$ is a diffeomorphism. Using the chain rule, $\mathbf{P}$-a.s. for all t and x,

$$\nabla X^{-1}_t(x) = \nabla X_t(X^{-1}_t(x))^{-1}. \tag{2.3.17}$$

Since, by Lemma 2.3.5, $\mathbf{P}$-a.s. $[\nabla X_\cdot(\cdot)]^{-1} \in D([0,T];C^{\beta'-1}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d))$ and we know that $\mathbf{P}$-a.s. for all t, $X^{-1}_t(\cdot)$ is differentiable, it follows from (2.3.17) that $\mathbf{P}$-a.s.

$$\nabla X_\cdot(X^{-1}_\cdot(\cdot))^{-1} \in D([0,T];C^{(\beta'-1)\wedge 1}_{\mathrm{loc}}(\mathbf{R}^d;\mathbf{R}^d)).$$

We proceed inductively to complete the proof. Making the obvious changes in the proof of part (1), we obtain part (2). $\square$

We conclude with a derivation of Hölder moment estimates of the inverse flow $X^{-1}_t(\tau,x)$, which will complete the proof of Theorem 2.2.1.

Proof of Theorem 2.2.1. (1) Fix a stopping time $\tau \leq T$ and write $X_t(\tau,x) = X_t(x)$. Fix $\epsilon > 0$. First, let us assume that $[\beta]^- = 1$. Set $J_t(x) = |\det \nabla X_t(x)|$. It is clear from (2.3.10) that for all $p \geq 2$ and x, there is a constant $N = N(d,p,N_0,T)$ such that

$$\mathbf{E}[\sup_{t\leq T} |J_t(x)|^p] \leq N. \tag{2.3.18}$$

By the mean value theorem, for all x and y and $\bar{p} \in \mathbf{R}$, we have

$$|r_1(x)^{\bar{p}} - r_1(y)^{\bar{p}}| \leq |\bar{p}|(r_1(x)^{\bar{p}-1} + r_1(y)^{\bar{p}-1})|x-y|. \tag{2.3.19}$$

Using the change of variable $(\bar{x},\bar{y}) = (X^{-1}_t(x), X^{-1}_t(y))$, Fatou's lemma, Fubini's theorem, Hölder's inequality, and the inequalities (2.3.3), (2.3.18), (2.3.19), (2.3.2), and (2.3.4), for all $\delta \in (0,1]$ and $p > \frac{d}{\epsilon}$, we conclude that there is a constant $N = N(d,p,N_0,T,\delta,\eta,N_\kappa,\epsilon)$

such that

$$\mathbf{E}\sup_{t\le T}\int_{\mathbf{R}^d}|r_1(x)^{-(1+\epsilon)}X_t^{-1}(x)|^p dx \le \int_{\mathbf{R}^d}|\bar{x}|^p\mathbf{E}\sup_{t\le T}[r_1(X_t(\bar{x}))^{-p(1+\epsilon)}J_t(\bar{x})]d\bar{x}$$
$$\le N\mathbf{E}\int_{\mathbf{R}^d} r_1(\bar{x})^{-p\epsilon}d\bar{x}\le N$$

and

$$\mathbf{E}\sup_{t\le T}\int_{|x-y|<1}\frac{|r_1^{-(1+\epsilon)}(x)X_t^{-1}(x)-r_1^{-(1+\epsilon)}(y)X_t^{-1}(y)|^p}{|x-y|^{2d+\delta p}}dxdy$$
$$\le\int_{|\bar{x}-\bar{y}|<1}\mathbf{E}\sup_{t\le T}\left[\frac{r_1^{-p(1+\epsilon)}(X_t(\bar{x}))|\bar{x}-\bar{y}|^pJ_t(\bar{x})J_t(\bar{y})}{|X_t(\bar{x})-X_t(\bar{y})|^{2d+\delta p}}\right]d\bar{x}d\bar{y}$$
$$+\int_{|\bar{x}-\bar{y}|<1}\mathbf{E}\sup_{t\le T}\left[\frac{|\bar{y}|^p|r_1^{-(1+\epsilon)}(X_t(\bar{x}))-r_1^{-(1+\epsilon)}(X_t(\bar{y}))|^pJ_t(\bar{x})J_t(\bar{y})}{|X_t(\bar{x})-X_t(\bar{y})|^{2d+\delta p}}\right]d\bar{x}d\bar{y}$$
$$\le N\int_{|\bar{x}-\bar{y}|<1}\frac{r_1(\bar{x})^{-p(1+\epsilon)}}{|\bar{x}-\bar{y}|^{2d-(1-\delta)p}}d\bar{x}d\bar{y}+N\int_{|\bar{x}-\bar{y}|<1}\frac{r_1(\bar{x})^{-p(1+\epsilon)}+r_1(\bar{y})^{-p(1+\epsilon)}}{|\bar{x}-\bar{y}|^{2d-(1-\delta)p}}d\bar{x}d\bar{y}\le N.$$

Similarly, making use of the inequalities (2.3.3), (2.3.18), (2.3.19), (2.3.2), (2.3.4), (2.3.14), and (2.3.15), for all $p>\frac{d}{\epsilon}\vee\frac{d}{\beta-\beta'}\vee\frac{d}{2-\beta'}$, we get

$$\mathbf{E}\sup_{t\le T}\int_{\mathbf{R}^d}|r^{-\epsilon}(x)\nabla X_t^{-1}(x)|^pdx\le\int_{\mathbf{R}^d}\mathbf{E}\sup_{t\le T}[r_1(X_t(\bar{x}))^{-p\epsilon}|[\nabla X_t(\bar{x})]^{-1}|^pJ_t(\bar{x})]d\bar{x}$$
$$\le N\mathbf{E}\int_{\mathbf{R}^d}r_1(\bar{x})^{-p\epsilon}d\bar{x}\le N$$

and

$$\mathbf{E}\sup_{t\le T}\int_{|x-y|<1}\frac{|r_1^{-\epsilon}(x)\nabla X_t^{-1}(x)-r_1^{-\epsilon}(y)\nabla X_t^{-1}(y)|^p}{|x-y|^{2d+(\beta'-1)p}}dxdy$$
$$\le\int_{|\bar{x}-\bar{y}|<1}\mathbf{E}\sup_{t\le T}\left[\frac{|r_1^{-\epsilon}(X_t(\bar{x}))[\nabla X_t(\bar{x})]^{-1}-r_1^{-\epsilon}(X_t(\bar{y}))[\nabla X_t(\bar{y})]^{-1}|^pJ_t(\bar{x})J_t(\bar{y})}{|X_t(\bar{x})-X_t(\bar{y})|^{2d+(\beta'-1)p}}\right]d\bar{x}d\bar{y}$$
$$\le N\int_{|\bar{x}-\bar{y}|<1}\frac{r_1(\bar{x})^{-p\epsilon}}{|\bar{x}-\bar{y}|^{2d-(\beta-\beta')p}}d\bar{x}d\bar{y}+N\int_{|\bar{x}-\bar{y}|<1}\frac{r_1(\bar{x})^{-p\epsilon}+r_1(\bar{y})^{-p\epsilon}}{|\bar{x}-\bar{y}|^{2d-(2-\beta')p}}d\bar{x}d\bar{y}\le N,$$

where $N=N(d,p,N_0,T,\beta',\eta,N_\kappa,\epsilon)$ is a positive constant. Therefore, combining the above estimates and applying Corollary 2.5.2, we find that for all $p\ge 2$, there is f a constant $N=N(d,p,N_0,T,\beta',\eta,N_\kappa,\epsilon)$, such that

$$\mathbf{E}\left[\sup_{t\le T}|r_1^{-(1+\epsilon)}X_t^{-1}(\tau)|_0^p\right]+\mathbf{E}\left[\sup_{t\le T}|r_1^{-\epsilon}\nabla X_t^{-1}(\tau)|_{\beta'-1}^p\right]\le N.$$

It is well-known that the the inverse map $\mathfrak{I}$ on the set of invertible $d\times d$-dimensional matrices is infinitely differentiable and for all n, there is a constant $N = N(n,d)$ such that for all invertible matrices M, the nth derivative of $\Im$ evaluated at M, denoted $\Im^{(n)}(M)$, satisfies

$$\left|\Im^{(n)}(M)\right| \leq N|M^{-n-1}| \leq N\left|M^{-1}\right|^{n+1}.$$

We claim that for all n and every multi-index γ with $|\gamma| = n$, the components of $\partial^{\gamma} X_t^{-1}(x)$ are a polynomial in terms of the entries of $[\nabla X_t(X_t^{-1}(x))]^{-1}$ and $\partial^{\gamma'}\nabla X_t(X_t^{-1}(x))$ for all multi-indices γ' with $1 \leq |\gamma'| \leq n-1$. Assume that statement holds for some n. By the chain rule, for all ω, t, and x, we have

$$\nabla(\nabla X_t(X_t^{-1}(x))^{-1}) = \Im^{(1)}(\nabla X_t(X_t^{-1}(x)))\nabla^2 X_t(X_t^{-1}(x))\nabla X_t(X_t^{-1}(x))^{-1}$$

and for all multi-indices γ with $1 \leq |\gamma'| \leq n-1$, we have

$$\nabla(\partial^{\gamma'}\nabla X_t(X_t^{-1}(x))) = \partial^{\gamma'}\nabla^2 X_t(X_t^{-1}(x))\nabla X_t(X_t^{-1}(x))^{-1},$$

where $\nabla^2 X_t(X_t^{-1}(x))$ is the tensor of second-order derivatives of $X_t(\cdot)$ evaluated at $X_t^{-1}(x)$. This implies that for every multi-index γ with $|\gamma| = n+1$, the components of $\partial^{\gamma} X_t^{-1}(x)$ are a polynomial in terms of the entries of $\nabla X_t(X_t^{-1}(x))^{-1}$ and $\partial^{\gamma'}\nabla X_t(X_t^{-1}(x))$ for all multi-indices γ' with $1 \leq |\gamma'| \leq n$. By induction, the claim is true. Therefore, for $[\beta]^- \geq 2$, using (2.3.10) and (2.3.11), we obtain the moment estimates for the inverse flow in the almost exact same way we did for $[\beta]^- = 1$. Making the obvious changes in the proof of part (1), we obtain part (2). This completes the proof of Theorem 2.2.1. □

2.3.3 Strong limit of a sequence of flows: Proof of Theorem 2.2.3

Proof of Theorem 2.2.3. Let $\tau \leq T$ be a fixed stopping time and write $X_t(x) = X_t(\tau, x)$. For each n, let

$$Z_t^{(n)}(x) = X_t^{(n)}(x) - X_t(x), \ (t,x) \in [0,T] \times \mathbf{R}^d.$$

Throughout the proof we denote by $(\delta_n)_{n\geq 1}$ a deterministic sequence with $\delta_n \to 0$ as $n \to \infty$ that may change from line to line. Let $N = N(p, N_0, T)$ be a positive constant, which may change from line to line. By virtue of Theorem 2.1 in [Kun04] and (2.3.1), for all $p \geq 2$ and t, x and n, we have

$$\mathbf{E}\left[\sup_{s\leq t}|Z_s^{(n)}(x)|^p\right] \leq N\mathbf{E}\int_{]0,t]}|Z_s^{(n)}(x)|^p ds + N\delta_n r_1(x)^p.$$

Since the right-hand-side is finite by (2.3.1), applying Gronwall's lemma we find that for all x and n,

$$\mathbf{E}\left[\sup_{t\le T}|Z_t^{(n)}(x)|^p\right] \le N\delta_n r_1(x)^p. \tag{2.3.20}$$

Similarly, by (2.3.10), for all x and n, we have

$$\mathbf{E}\left[\sup_{t\le T}|\nabla Z_t^{(n)}(x)|^p\right] \le N\delta_n.$$

Using (2.3.10), for all x, y, and n, we obtain

$$\mathbf{E}\left[\sup_{t\le T}|Z_t^{(n)}(x) - Z^{(n)}(y)|^p\right] \le |x-y|^p \mathbf{E}\sup_{t\le T}\int_0^1 |\nabla Z_t^{(n)}(y+\theta(x-y))|^p d\theta \le N|x-y|^p.$$

It follows immediately from (2.3.11) that for all x, y, and n,

$$\mathbf{E}\left[\sup_{t\le T}]|\nabla Z_t^{(n)}(x) - \nabla Z_t^{(n)}(y)|^p\right] \le N|x-y|^{(\beta-1)\vee 1}.$$

Thus, by Corollary 2.5.4, we have

$$\lim_{n\to\infty}\left(\mathbf{E}\left[\sup_{t\le T}|r_1^{-(1+\epsilon)}X_t^{(n)} - r_1^{-(1+\epsilon)}X_t|_0^p\right] + \mathbf{E}\left[\sup_{t\le T}|r_1^{-\epsilon}\nabla X_t^{(n)} - r_1^{-\epsilon}\nabla X_t|_0^p\right]\right) = 0. \tag{2.3.21}$$

Owing to a standard interpolation inequality for Hölder spaces (see, e.g. Lemma 6.32 in [GT01]), for all $\delta \in (0,1)$ and $\bar\beta \in (\beta', \beta)$, there is a constant $N(\delta)$ such that

$$\begin{aligned}\mathbf{E}\left[\sup_{t\le T}|r_1^{-\epsilon}\nabla X_t^{(n)} - r_1^{-\epsilon}\nabla X_t|_{\beta'-1}^p\right] &\le \delta\mathbf{E}\left[\sup_{t\le T}|r_1^{-\epsilon}\nabla X_t^{(n)} - \nabla X_t|_{\bar\beta-1}^p\right]\\ &\quad + C_\delta\mathbf{E}\left[\sup_{t\le T}|r_1^{-\epsilon}\nabla X_t^{(n)} - \nabla X_t|_0^p\right],\end{aligned}$$

and hence since

$$\sup_{n\in\mathbf{N}}\mathbf{E}\left[\sup_{t\le T}|r_1^{\varepsilon}\nabla X^{(n)}|_{\bar\beta-1}^p\right] + \mathbf{E}\left[\sup_{t\le T}|r_1^{\varepsilon}\nabla X_t|_{\bar\beta-1}^p\right] < \infty,$$

we have

$$\lim_{n\to\infty}\mathbf{E}[\sup_{t\le T}|r_1^{-\epsilon}\nabla X_t^{(n)} - r_1^{-\epsilon}\nabla X_t|_{\beta'-1}^p] = 0.$$

By Theorem 2.2.1, Corollary 2.5.4, and the interpolation inequality for Hölder spaces used above, in order to show

$$\lim_{n\to\infty}\mathbf{E}\left[\sup_{t\le T}|r_1^{-(1+\epsilon)}X_t^{(n);-1}(\tau) - r_1^{-(1+\epsilon)}X_t^{-1}(\tau)|_0^p\right] = 0$$

$$\lim_{n\to\infty} \mathbf{E}\left[\sup_{t\le T} |r_1^{-\epsilon}\nabla X_t^{(n);-1}(\tau) - r_1^{-\epsilon}\nabla X_t^{-1}(\tau)|_{\beta'-1}^p\right] = 0,$$

it suffices to show that for all x,

$$d\mathbf{P} - \lim_{n\to\infty} \sup_{t\le T} |X_t^{(n);-1}(x) - X_t^{-1}(x)| = 0 \tag{2.3.22}$$

and

$$d\mathbf{P} - \lim_{n\to\infty} \sup_{t\le T} |\nabla X_t^{(n);-1}(x) - \nabla X_t^{-1}(x)| = 0. \tag{2.3.23}$$

For each n, define

$$\Theta_t^{(n)}(x) = r_1(X_t^{(n)}(x))^{-1} - r_1(X_t(x))^{-1},\ (t,x) \in [0,T]\times\mathbf{R}^d.$$

For all ω, t, x, and n, we have

$$|\Theta_t^{(n)}(x)| \le r_1(X_t^{(n)}(x))^{-1} r_1(X_t(x))^{-1} |Z_t^{(n)}(x)|,$$

and hence using Hölder's inequality, (2.3.4), and (2.3.20), we obtain that for all $p \ge 2$, x, there is a constant $N = N(p, N_0, T, \eta, N_\kappa)$ such that for all n,

$$\mathbf{E}\left[\sup_{t\le T} |\Theta_t^{(n)}(x)|^p\right] \le N r_1(x)^{-p}\delta_n,$$

where $N = N(p, N_0, T, \eta, N_\kappa)$ is a constant. Furthermore, since

$$|\nabla\Theta_t^{(n)}(x)| \le r_1(X_t^{(n)}(x))^{-2}|\nabla X_t^{(n)}(x)| + r_1(X_t(x))^{-2}|\nabla X_t^{(n)}(x)|,$$

for all ω, t, x, and n, applying (2.3.4) and (2.3.10), for all $p \ge 2$, x, and n, we get

$$\mathbf{E}\left[\sup_{t\le T} |r_1(x)\Theta_t^{(n)}(x) - r_1(y)\Theta_t^{(n)}(y)|^p\right] \le N|x-y|^p.$$

Then owing to Corollary 2.5.4, for all $p \ge 2$,

$$\lim_{n\to\infty} \mathbf{E}\left[\sup_{t\le T} |\Theta_t^{(n)}|_0^p\right] = 0. \tag{2.3.24}$$

We claim that for all $R > 0$,

$$d\mathbf{P} - \lim_{n\to\infty} E(n,R) := d\mathbf{P} - \lim_{n\to\infty} \sup_{t\le T} |X_t^{(n);-1} - X_t^{-1}|_{0;\{|x|\le R\}} = 0. \tag{2.3.25}$$

Fix $R > 0$. It is enough to show that every subsequence of $E(n) = E(n,R)$ has a sub-subsequence converging to 0, $\mathbf{P}$-a.s.. Owing to (2.3.21) and (2.3.24), for a given subsequence $(E(n_k))$, we can always find sub-subsequence (still denoted $(E(n_k))$ to avoid double indices) such that $\mathbf{P}$-a.s.,

$$\lim_{k\to\infty}\sup_{t\le T}|X_t^{(n_k)}-X_t|_{\beta';\{|x|\le\bar{R}\}}=0,\quad\forall\ \bar{R}>0,\tag{2.3.26}$$

and

$$\lim_{k\to\infty}\sup_{t\le T}|r_1(X_t^{(n_k)}(x))^{-1}-r_1(X_t(x))^{-1}|_0=0.$$

Fix an ω for which both limits are zero. We will prove that

$$\lim_{k\to\infty}\sup_{t\le T}|X_t^{(n_k);-1}(\omega)-X_t^{-1}(\omega)|_{0;\{|x|\le R\}}=0.\tag{2.3.27}$$

Suppose, by contradiction, that (2.3.27) is not true. Then there exists an $\varepsilon>0$ and a subsequence of (n_k) (still denoted (n_k)) such that $t_{n_k}\to t-$ (or $t_{n_k}\to t+$) and $x_{n_k}\to x$ as $k\to\infty$ with $\left|x_{n_k}\right|\le R$ such that (dropping ω),

$$|X_{t_{n_k}}^{(n_k);-1}(x_{n_k})-X_{t_{n_k}}^{-1}(x_{n_k})|\ge\varepsilon.\tag{2.3.28}$$

Arguing by contradiction and using (2.3.3), we have

$$\sup_{k\in\mathbf{N}}|X_{t_{n_k}}^{(n_k);-1}(x_{n_k})|<\infty.\tag{2.3.29}$$

Applying (2.3.29), (2.3.26), and the fact that $X_\cdot(\cdot),X_\cdot^{-1}(\cdot)\in D([0,T];C_{\mathrm{loc}}^{\beta'}(\mathbf{R}^d;\mathbf{R}^d))$, we obtain

$$\begin{aligned}\lim_{k\to\infty}\left(X_{t-}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k}))-X_{t-}(X_{t_{n_k}}^{-1}(x_{n_k}))\right)&=\lim_{k\to\infty}\left(X_{t-}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k}))-x_{n_k}\right)\\&=\lim_{k\to\infty}\left(X_{t-}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k}))-X_{t_{n_k}}^{(n_k)}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k}))\right)\\&=\lim_{k\to\infty}\left(X_{t-}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k}))-X_{t_{n_k}}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k}))\right)\\&+\lim_{k\to\infty}\left(X_{t_{n_k}}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k}))-X_{t_{n_k}}^{(n_k)}(X_{t_{n_k}}^{(n_k);-1}(x_{n_k}))\right)=0,\end{aligned}$$

which contradicts (2.3.28), and hence proves (2.3.27), (2.3.25), and (2.3.22). For each n, define

$$\bar{U}_t^{(n)}=\bar{U}^{(n)}(t,x)=\nabla X_t^{(n)}(x)^{-1}\quad\text{and}\quad\bar{U}(t)=\bar{U}(t,x)=\nabla X_t(x)^{-1},\quad(t,x)\in[0,T]\times\mathbf{R}^d.$$

Using (2.3.14) and (2.3.15) and repeating the arguments given above, for all $p \geq 2$, we get

$$\lim_{n\to\infty} \mathbf{E}\left[\sup_{t\leq T} |r_1^{-\epsilon}\bar{U}_t^{(n)} - r_1^{-\epsilon}\bar{U}_t|_{\beta'-1}^p\right] = 0.$$

Then (2.3.3) and (2.3.25) imply that for all $R > 0$,

$$\begin{aligned} & d\mathbf{P} - \lim_{n\to\infty} \sup_{t\leq T} |\nabla X_t^{(n);-1}(x) - \nabla X_t^{-1}(x)|_{0;\{|x|\leq R\}} \\ &= d\mathbf{P} - \lim_{n\to\infty} \sup_{t\leq T} |\nabla X_t^{(n)}(X_t^{(n);-1}(x))^{-1} - \nabla X_t(X_t^{-1}(x))^{-1}|_{0;\{|x|\leq R\}} = 0, \end{aligned}$$

which yields (2.3.23) and completes the proof. □

2.4 Classical solutions of degenerate SPDEs: Proof of Theorem 2.2.4

Proof of Theorem 2.2.4. Fix a stopping time $\tau \leq T$. By virtue of Theorem 2.2.1, we only need to show that $Y^{-1}(\tau) = Y_t^{-1}(\tau, x)$ solves (2.1.2) and is the unique solution. Suppose we have shown that $Y^{-1}(s, x)$, $s \in [0, T]$, solves (2.1.2) (i.e. where $\tau = s$ is deterministic). It is then a routine argument to conclude that $Y^{-1}(\tau')$ solves (2.1.2) for finite-valued stopping times $\tau' \leq T$. We can then use an approximation argument as in the proof of Proposition 2.3.3 to show that $Y^{-1}(\tau) = Y_t^{-1}(\tau, x)$ solves (2.1.2). Thus, it suffices to take τ deterministic. Let $u_t(x) = u_t(s, x) = Y_t^{-1}(s, x)$, $(s, t, x) \in [0, T]^2 \times \mathbf{R}^d$. Fix $(s, t, x) \in [0, T]^2 \times \mathbf{R}^d$ with $s < t$ and write $Y_t(x) = Y_t(s, x)$. Let $((t_n^M)_{0\leq n\leq M})_{1\leq M\leq\infty}$ be a sequence of partitions of the interval $[s, t]$ such that for all $M > 0$, $(t_n^M)_{0\leq n\leq M}$ has mesh size $(t - s)/M$. Fix M and set $(t_n)_{0\leq n\leq M} = (t_n^M)_{0\leq n\leq M}$. Immediately, we obtain

$$u_t(x) - x = \sum_{n=0}^{M-1} (u_{t_{n+1}}(x) - u_{t_n}(x)). \tag{2.4.1}$$

We will use Taylor's theorem to expand each term in the sum on the right-hand-side of (2.4.1). By Taylor's theorem, for all n and y, we have

$$\begin{aligned} & u_{t_{n+1}}(Y_{t_{n+1}}(y)) - u_{t_n}(Y_{t_{n+1}}(y)) = y - u_{t_n}(Y_{t_{n+1}}(y)) = u_{t_n}(Y_{t_n}(y)) - u_{t_n}(Y_{t_{n+1}}(y)) \\ &= \nabla u_{t_n}(Y_{t_n}(y))(Y_{t_n}(y) - Y_{t_{n+1}}(y)) - (Y_{t_n}(y) - Y_{t_{n+1}}(y))^* \Theta_n(Y_{t_n}(y))(Y_{t_n}(y) - Y_{t_{n+1}}(y)), \end{aligned} \tag{2.4.2}$$

where

$$\Theta_n^{ij}(z) = \int_0^1 (1-\theta)\partial_{ij} u_{t_n}\left(z + \theta(Y_{t_{n+1}}(Y_{t_n}^{-1}(z)) - z)\right) d\theta.$$

Since for all n, $Y_{t_{n+1}}(s,x) = Y_{t_{n+1}}(t_n, Y_{t_n}(s,x))$, we have

$$Y_{t_{n+1}}(Y_{t_n}^{-1}(x)) = Y_{t_{n+1}}(t_n, x)$$

and hence substituting $y = Y_{t_n}^{-1}(x)$ into (2.4.2), for all n, we get

$$u_{t_{n+1}}(x) - u_{t_n}(x) = A_n + B_n, \tag{2.4.3}$$

where

$$A_n := \nabla u_{t_n}(x)(x - Y_{t_{n+1}}(t_n,x)) - (x - Y_{t_{n+1}}(t_n,x))^* \Theta_n^{ij}(x)(x - Y_{t_{n+1}}(t_n,x))$$

and

$$B_n := (u_{t_{n+1}}(x) - u_{t_n}(x)) - (u_{t_{n+1}}(Y_{t_{n+1}}(t_n,x)) - u_{t_n}(Y_{t_{n+1}}(t_n,x))).$$

Applying Taylor's theorem once more, for all n, we obtain

$$B_n = C_n + D_n, \tag{2.4.4}$$

where

$$C_n := (\nabla u_{t_{n+1}}(x) - \nabla u_{t_n}(x))(x - Y_{t_{n+1}}(t_n,x)),$$
$$D_n := -(x - Y_{t_{n+1}}(t_n,x))^* \tilde{\Theta}_n(x)(x - Y_{t_{i+1}}(t_i,x))),$$

and

$$\tilde{\Theta}_n(x)^{ij} := \int_0^1 (1-\theta)\partial_{ij}(u_{t_{n+1}} - u_{t_n})(x + \theta(Y_{t_{n+1}}(t_n,x) - x))d\theta.$$

Thus, combining (2.4.1), (2.4.3), and (2.4.4), $\mathbf{P}$-a.s. we have

$$u_t(x) - x = \sum_{n=0}^{M-1}(A_n + C_n + D_n). \tag{2.4.5}$$

Now, we will derive the limit of the right-hand-side of (2.4.5).

Claim 2.4.1. *(1)*

$$d\mathbf{P} - \lim_{M\to\infty} \sum_{n=0}^{M-1} A_n = -\int_{]s,t]} [\frac{1}{2}\sigma_r^{i\varrho}(x)\sigma_r^{j\varrho}(x)\partial_{ij}u_r(x) + b_r^i(x)\partial_i u_r(x)]dr$$
$$- \int_{]s,t]} \sigma_r^{i\varrho}(x)\partial_i u_r(x) dw_r^{\varrho};$$

(2) $d\mathbf{P} - \lim_{M\to\infty} \sum_{n=0}^{M-1} D_n = 0$;
(3) $d\mathbf{P} - \lim_{M\to\infty} \sum_{n=0}^{M-1} C_n = \int_{]s,t]} \sigma_r^{j\varrho}(x)\partial_j\sigma_r^{i\varrho}(x)\partial_i u_r(x)dr + \int_{]s,t]} \sigma_r^{i\varrho}(x)\sigma_r^{j\varrho}(x)\partial_{ij}u_r(x)dr.$

Proof of Claim 2.4.1. (1) For all n, we have

$$\nabla u_{t_n}(x)\,(x - Y_{t_{n+1}}(t_n,x)) = -\int_{]t_n,t_{n+1}]} b^i_r(x)\partial_i u_{t_n}(x)dr - \int_{]t_n,t_{n+1}]} \sigma^{i\varrho}_r(x)\partial_i u_{t_n}(x)dw^\varrho_r + R^{(1)}_n + R^{(2)}_n,$$

where

$$R^{(1)}_n := \int_{]t_n,t_{n+1}]} \left(b^i_r(x) - b^i_r(Y_r(t_n,x))\right)\partial_i u_{t_n}(x)dr$$

and

$$R^{(2)}_n := \int_{]t_n,t_{n+1}]} [\sigma^{i\varrho}_r(x) - \sigma^{i\varrho}_r(Y_r(t_n,x))]\partial_i u_{t_n}(x)dw^\varrho_r.$$

Since b and σ are Lipschitz, there is a constant $N = N(N_0,T)$ such that

$$\sum_{n=0}^{M-1}\left|R^1_n\right| \le N\sup_{s\le r\le t}|\nabla u_r(x)|\sup_{|r_1-r_2|\le\frac{t}{M}}|x - Y_{r_1}(r_2,x)|$$

and

$$\int_{]s,t]}\left|\sum_{n=0}^{M-1}\mathbf{1}_{]t_n,t_{n+1}]}(r)\left(\sigma^{i\cdot}_r(x) - \sigma^{i\cdot}_r(Y_r(t_n,x))\right)\partial_i u_{t_n}(x)\right|^2 ds \le N\sup_{s\le r\le t}|\nabla u_r(x)|^2\sup_{|r_1-r_2|\le\frac{t}{M}}|x - Y_{r_1}(r_2,x)|^2.$$

Owing to the joint continuity of $Y_t(s,x)$ in s and t and the dominated convergence theorem for stochastic integrals, we obtain

$$d\mathbf{P} - \lim_{M\to\infty}\sum_{n=0}^{M-1}(R^{(1)}_n + R^{(2)}_n) = 0. \tag{2.4.6}$$

In a similar way, this time using the continuity of $\nabla u_t(x)$ in t and the linear growth of b and σ, we find

$$d\mathbf{P} - \lim_{M\to\infty}\sum_{n=0}^{M-1}\left(-\int_{]t_n,t_{n+1}]} b^i_r(x)\partial_i u_{t_n}(x)dr - \int_{]t_n,t_{n+1}]}\sigma^{i\varrho}_r(x)\partial_i u_{t_n}(x)dw^\varrho_r\right)$$
$$= -\int_{]s,t]} b_r(x)\partial_i u_r(x)dr - \int_{]s,t]}\sigma^\varrho_r(x)\partial_i u_r(x)dw^\varrho_r.$$

For all n, we have

$$-(x - Y_{t_{n+1}}(t_n,x))^*\Theta_n(x)(x - Y_{t_{n+1}}(t_n,x)) = S^{(1)}_n + S^{(2)}_n,$$

where $S_n^{(1)}(t,x)$ has only $drdr$ and $drdw_r^\varrho$ terms and where

$$S_n^{(2)} := -\frac{1}{2}\left(\int_{]t_n,t_{n+1}]}\sigma_r^{i\varrho}(Y_r(t_n,x))dw_r^\varrho\right)\partial_{ij}u_{t_n}(x)\left(\int_{]t_n,t_{n+1}]}\sigma_r^{j\varrho}(Y_r(t_n,x))dw_r^\varrho\right)$$
$$-\left(\int_{]t_n,t_{n+1}]}\sigma_r^{i\varrho}(Y_r(t_n,x))dw_r^\varrho\right)\left(\Theta_n^{ij}(x)-\frac{1}{2}\partial_{ij}u_{t_n}(x)\right)\left(\int_{]t_n,t_{n+1}]}\sigma_r^{j\varrho}(Y_r(t_n,x))dw_r^\varrho\right).$$

Since

$$\left|\Theta_n^{ij}(x)-\frac{1}{2}\partial_{ij}u_{t_n}(x)\right| = \left|\int_0^1(1-\theta)(\partial_{ij}u_{t_n}(x+\theta(Y_{t_{n+1}}(t_n,x)-x))-\partial_{ij}u_{t_n}(x))d\theta\right|$$
$$\le N\sup_{|r_1-r_2|\le\frac{t}{M},\theta\in(0,1)}|\partial_{ij}u_{r_1}(x+\theta(Y_{r_2}(r_1,x)-x))-\partial_{ij}u_{r_1}(x))|,$$

proceeding as in the derivation of (2.4.6) and using the joint continuity of $\partial_{ij}u_t(x)$ in t and x, the continuity of $Y_t(s,x)$ in s and t, the explicit form of the quadratic variation of the stochastic integral (i.e. part (5) of Chapter 2, Section 2, Theorem 2 in [LS89]), and the stochastic dominated convergence theorem, we obtain

$$d\mathbf{P}-\lim_{M\to\infty}\sum_{n=0}^{M-1}S_n^{(2)} = -\frac{1}{2}\int_{]0,t]}\sigma_r^{i\varrho}(x)\sigma_r^{j\varrho}(x)\partial_{ij}u_r(x)dr.$$

Similarly, making use of the properties stated in Theorem 2.2.1(2), we have

$$d\mathbf{P}-\lim_{M\to\infty}\sum_{n=0}^{M-1}S_n^{(1)} = 0,$$

which completes the proof of part (1). The proof of part (2) is similar to the proof of part (1), so we proceed to the proof of part (3). We know that for all n, $Y_{t_{n+1}}(x) = Y_{t_{n+1}}(t_n, Y_{t_n}(x))$. Thus, for all n, we have $u_{t_{n+1}}(x) = u_{t_n}(Y_{t_{n+1}}^{-1}(t_n,x))$, and hence by the chain rule,

$$\nabla u_{t_{n+1}}(x) = \nabla u_{t_n}(Y_{t_{n+1}}^{-1}(t_n,x))\nabla Y_{t_{n+1}}^{-1}(t_n,x). \tag{2.4.7}$$

By (2.4.7) and Taylor's theorem, for all n, we have

$$C_n = (\nabla u_{t_{n+1}}(x)-\nabla u_{t_n}(x))(x-Y_{t_{n+1}}(t_n,x))$$
$$= \nabla u_{t_n}(Y_{t_{n+1}}^{-1}(t_n,x))(\nabla Y_{t_{n+1}}^{-1}(t_n,x)-I_d)(x-Y_{t_{n+1}}(t_n,x))$$
$$+(Y_{t_{n+1}}^{-1}(t_n,x)-x)^*\tilde\Theta_n(x)(x-Y_{t_{n+1}}(t_n,x)) =: E_n+F_n,$$

where

$$\tilde\Theta_n^{ij}(x) := \int_0^1\partial_{ij}u_{t_n}(x+\theta(Y_{t_{n+1}}^{-1}(t_n,x)-x))d\theta.$$

Using Itô's formula, for all n, we get (see, also, Lemma 3.12 in [Kun04]),

$$\nabla Y_{t_{n+1}}(t_n,x)^{-1} = I_d - \int_{]t_n,t_{n+1}]} \nabla Y_r(t_n,x)^{-1}\nabla\sigma_r^{\varrho}(Y_r(t_n,x))dw_r^{\varrho} \\ + \int_{]t_n,t_{n+1}]} \nabla Y_r(t_n,x)^{-1}\left(\nabla\sigma_r^{\varrho}(Y_r(t_n,y))\nabla\sigma_r^{\varrho}(Y_r(t_n,x)) - \nabla b_r(Y_r(t_n,x))\right)dr,$$

and hence

$$\nabla Y_{t_{n+1}}^{-1}(t_n) - I_d = \nabla Y_{t_{n+1}}^{-1}(t_n, Y_{t_{n+1}}^{-1}(t_n,x)) - I_d =: G_{t_n,t_{n+1}}^{(1)}(Y_{t_{n+1}}^{-1}(t_n,x)) + G_{t_n,t_{n+1}}^{(2)}(Y_{t_{n+1}}^{-1}(t_n,x)),$$

where for $y \in \mathbf{R}^d$,

$$G_{t_n,t_{n+1}}^{(1)}(y) := \int_{]t_n,t_{n+1}]} \nabla Y_r(t_n,z)^{-1}\left(\nabla\sigma_r^{\varrho}(Y_r(t_n,y))\nabla\sigma_r^{\varrho}(Y_r(t_n,y)) - \nabla b_r(Y_z(t_n,y))\right)dr$$

and

$$G_{t_n,t_{n+1}}^{(2)}(z) := -\int_{]t_n,t_{n+1}]} \nabla Y_r(t_n,y)^{-1}\nabla\sigma_r^{\varrho}(Y_r(t_n,y))dw_r^{\varrho}.$$

By the Burkholder-Davis-Gundy inequality, Hölder's inequality, and the inequalities (2.3.2), (2.3.14), and (2.3.15), for all $p \geq 2$, there is a constant $N = N(p,d,N_0,T)$ such that for all x_1 and x_2,

$$\mathbf{E}\left[|G_{t_n,t_{n+1}}^{(2)}(x_1)|^p\right] \leq NM^{-p/2+1}\int_{]t_n,t_{n+1}]} \mathbf{E}\left[|\nabla Y_r(t_n,x_1)^{-1}|^p|\nabla\sigma_r(Y_r(t_n,x_1)|^p\right]dr \leq NM^{-p/2}$$

and

$$\mathbf{E}\left[|G_{t_n,t_{n+1}}^{(2)}(x_1) - G_{t_n,t_{n+1}}^{(2)}(x_2)|^p\right] \leq NM^{-p/2+1}\int_{]t_n,t_{n+1}]} \mathbf{E}\left[|\nabla Y_r(t_n,x_1)^{-1} - \nabla Y_r(t_n,x_2)^{-1}|^p\right]dr \\ + NM^{-p/2+1}\int_{]t_n,t_{n+1}]} \left(\mathbf{E}\left[|\nabla Y_r(t_n,x_1)^{-1}|^{2p}\right]\right)^{1/2}\left(\mathbf{E}\left[|Y_r(t_n,x_1) - Y_r(t_n,x_2)|^{2p}\right]\right)^{1/2}dr \\ \leq NM^{-p/2}|x-y|^p.$$

Thus, by Corollary 2.5.3, we obtain that for all $p \geq 2$, $\epsilon > 0$, and $\delta < 1$, there is a constant $N = N(p,d,\delta,N_0,T)$ such that

$$\mathbf{E}\left[|r^{-\epsilon}G_{t_n,t_{n+1}}^{(2)}|_{\delta}^p\right] \leq NM^{-p/2}. \tag{2.4.8}$$

For all n, we have

$$E_n = \nabla u_{t_n}(Y_{t_{n+1}}^{-1}(t_n,x))G_{t_n,t_{n+1}}^{(1)}(Y_{t_{n+1}}^{-1}(t_n,x))(x - Y_{t_{n+1}}(t_n,x)) \\ + \nabla u_{t_n}(x)G_{t_n,t_{n+1}}^{(2)}(x)(x - Y_{t_{n+1}}(t_n,x))$$

$$+ \nabla u_{t_n}(Y^{-1}_{t_{n+1}}(t_n, x))(G^{(2)}_{t_n,t_{n+1}}(Y^{-1}_{t_{n+1}}(t_n, x)) - G^{(2)}_{t_n,t_{n+1}}(x))(x - Y_{t_{n+1}}(t_n, x))$$
$$+ (\nabla u_{t_n}(Y^{-1}_{t_{n+1}}(t_n, x) - \nabla u_{t_n}(x))G^{(2)}_{t_n,t_{n+1}}(x)(x - Y_{t_{n+1}}(t_n, x))$$

One can easily check that

$$d\mathbf{P} - \lim_{M\to\infty} \sum_{n=0}^{M-1} \nabla u_{t_n}(Y^{-1}_{t_{n+1}}(t_n, x))G^{(1)}_{t_n,t_{n+1}}(Y^{-1}_{t_{n+1}}(t_n, x))(x - Y_{t_{n+1}}(t_n, x)) = 0. \tag{2.4.9}$$

Since $\nabla u_t(x)$ is jointly continuous in t and x and $Y^{-1}_t(s, x)$ is jointly in s and t, we have

$$d\mathbf{P} - \lim_{M\to\infty} \sup_n |\nabla u_{t_n^M}(Y^{-1}_{t^M_{n+1}}(t^M_n, x)) - \nabla u_{t_n^M}(x)| = 0.$$

Moreover, using Hölder's inequality, (2.4.8), and (2.3.1), we get

$$\sup_M \mathbf{E} \sum_{n=0}^{M-1} |G^{(2)}_{t_n,t_{n+1}}(x)|x - Y_{t_{n+1}}(t_n, x)| < \infty,$$

and hence

$$d\mathbf{P} - \lim_{M\to\infty} \sum_{n=0}^{M-1} (\nabla u_{t_n}(Y^{-1}_{t_{n+1}}(t_n, x)) - \nabla u_{t_n}(x))G^{(2)}_{t_n,t_{n+1}}(x)(x - Y_{t_{n+1}}(t_n, x)) = 0. \tag{2.4.10}$$

We claim that

$$d\mathbf{P} - \lim_{M\to\infty} \sum_{n=0}^{M-1} \nabla u_{t_n}(Y^{-1}_{t_{n+1}}(t_n, x))\left(G^{(2)}_{t_n,t_{n+1}}(Y^{-1}_{t_{n+1}}(t_n, x)) - G^{(2)}_{t_n,t_{n+1}}(x)\right)(x - Y_{t_{n+1}}(t_n, x)) = 0. \tag{2.4.11}$$

Set

$$J^M = \sum_{n=0}^{M-1} |G^{(2)}_{t_n,t_{n+1}}(Y^{-1}_{t_{n+1}}(t_n, x)) - G^{(2)}_{t_n,t_{n+1}}(x)|x - Y_{t_{n+1}}(t_n, x)|.$$

For all $\bar{\delta}, \epsilon \in (0, 1)$, we have

$$\mathbf{P}(J^M > \bar{\delta}) \le \mathbf{P}\left(J^M > \bar{\delta},\ \max_n |Y^{-1}_{t_{n+1}}(t_n, x) - x| \le \epsilon\right) + \mathbf{P}\left(\max_n |Y^{-1}_{t_{n+1}}(t_n, x) - x| > \epsilon\right).$$

By virtue of (2.4.8), there is a deterministic constant $N = N(x)$ independent of M such that for all $\omega \in V^M := \{\max_n |Y^{-1}_{t_{n+1}}(t_n, x) - x| \le \epsilon\}$,

$$J^M \le N\epsilon^{\delta} \sum_{n=0}^{M-1} [r_1^{-\epsilon} G^{(2)}_{t_n,t_{n+1}}]_{\delta}|x - Y_{t_{n+1}}(t_n, x)|,$$

which implies that

$$\mathbf{E}\mathbf{1}_{V^M} J^M \leq N\epsilon^{\delta}\mathbf{E}\sum_{n=0}^{M-1}\Big([r_1^{-\epsilon}G^{(2)}_{t_n,t_{n+1}}]^2_{\delta} + |x - Y_{t_{n+1}}(t_n,x)|^2\Big) \leq N\epsilon^{\delta}\sum_{n=0}^{M-1} M^{-1} \leq N\epsilon^{\delta}.$$

Applying Markov's inequality, we derive

$$\mathbf{P}(J^M| > \bar{\delta},\ \max_n |Y^{-1}_{t_{n+1}}(t_n,x) - x| \leq \epsilon) \leq N\frac{\epsilon^{\delta}}{\bar{\delta}},$$

and hence for all $\bar{\delta} > 0$,

$$\lim_{M\to\infty}\mathbf{P}(J^M > \bar{\delta}) = 0,$$

which yields (2.4.11). Owing to (2.4.9), (2.4.10), and (2.4.11) we have

$$d\mathbf{P} - \lim_{M\to\infty}\sum_{n=0}^{M-1} E_n = \lim_{M\to\infty}\sum_{n=0}^{M-1}\nabla u_{t_n}(x)G^{(2)}_{t_n,t_{n+1}}(x)(x - Y_{t_{n+1}}(t_n,x)).$$

Proceeding as in the proof of part (1) of the claim, we obtain

$$\begin{aligned}
&\lim_{M\to\infty}\sum_{n=0}^{M-1} K_n \\
&= \lim_{M\to\infty}\sum_{n=0}^{M-1}\nabla u_{t_n}(x)\int_{]t_n,t_{n+1}]}(\nabla Y_r(t_n,x)^{-1} - I_d)\nabla\sigma^{\varrho}_r(x)dW^{\varrho}_r\int_{]t_n,t_{n+1}]}\sigma^{\varrho}_r(x)dW^{\varrho}_r \\
&\quad + \lim_{M\to\infty}\sum_{n=0}^{M-1}\nabla u_{t_n}(x)\int_{]t_n,t_{n+1}]}\nabla\sigma^{\varrho}_r(x)dw^{\varrho}_r\int_{]t_n,t_{n+1}]}\sigma^{\varrho}_r(x)dw^{\varrho}_r \\
&\quad = \int_{]s,t]}\sigma^{j\varrho}_r(x)\partial_j\sigma^{i\varrho}_r(x)\partial_i u_r(x)dr. \qquad (2.4.12)
\end{aligned}$$

It is easy to check that for all n,

$$\begin{aligned}
F_n &= (Y^{-1}_{t_{n+1}}(t_n,x) - x)^*\tilde{\Theta}_n(x)(x - Y_{t_{n+1}}(t_n,x)) \\
&=: (G^{(3)}_{t_n,t_{n+1}}(Y^{-1}_{t_{n+1}}(t_n,x)) + G^{(4)}_{t_n,t_{n+1}}(Y^{-1}_{t_{n+1}}(t_n,x)))^*\tilde{\Theta}_n(x)(x - Y_{t_{n+1}}(t_n,x)),
\end{aligned}$$

where for $y \in \mathbf{R}^d$,

$$G^{(3)}_{t_n,t_{n+1}}(y) := -\int_{]t_n,t_{n+1}]} b_r(Y_r(t_n,y))dr, \quad G^{(4)}_{t_n,t_{n+1}}(y) := -\int_{]t_n,t_{n+1}]}\sigma^{\varrho}_r(Y_r(t_n,y))dw^{\varrho}_r.$$

Arguing as in the proof of (2.4.12), we get

$$d\mathbf{P} - \lim_{M\to\infty}\sum_{n=0}^{M-1} F_n = \int_{]s,t]} \sigma_r^{i\varrho}(x)\sigma_r^{j\varrho}(x)\partial_{ij}u_r(x)dt,$$

which completes the proof of the claim. □

By virtue of (2.4.5) and Claim 2.4.1, for all s and t with $s \le t$ and x, $\mathbf{P}$-a.s.

$$u_t(x) = x + \int_{]s,t]} \left(\frac{1}{2}\sigma_r^{i\varrho}(x)\sigma^{j\varrho}(x)\partial_{ij}u_r(x) - \hat{b}_t^i(x)\partial_i u_r(x)\right)dr - \int_{]s,t]} \sigma_r^{i\varrho}(x)\partial_i u_r(x)dw_r^\varrho. \tag{2.4.13}$$

Owing to Theorem 2.2.1, $u = u_t(x)$ has a modification that is jointly continuous in s and t and twice continuously differentiable in x. It is easy to check that the Lebesgue integral on the right-hand-side of (2.4.13) has a modification that is continuous in s, t, and x. Thus, the stochastic integral on the right-hand-side of (2.4.13) has a modification that is continuous in s, t, and x, and hence the equality in (2.4.13) holds $\mathbf{P}$-a.s. for all s and t with $s \le t$ and x. This proves that $Y^{-1}(\tau) = Y_t^{-1}(\tau, x)$ solves (2.1.2). However, if $u^1(\tau), u^2(\tau) \in \mathfrak{C}^{\beta'}_{\text{cts}}(\mathbf{R}^d;\mathbf{R}^d)$ are solutions of (2.1.2), then applying the Itô-Wentzell formula (see, e.g. Theorem 9 in Chapter 1, Section 4.8 in [Roz90]), we get that $\mathbf{P}$-a.s. for all t and x,

$$u_t^1(\tau, Y_t(\tau, x)) = x = u_t^2(\tau, Y_t(\tau, x)),$$

which implies that $\mathbf{P}$-a.s. for all t and x, $u^1(\tau) = Y_t^{-1}(\tau, x) = u^2(\tau)$. Thus, $Y^{-1}(\tau) = Y_t^{-1}(\tau, x)$ is the unique solution of (2.1.2) in $\mathfrak{C}^{\beta'}_{\text{cts}}(\mathbf{R}^d;\mathbf{R}^d)$. □

2.5 Appendix

Let V be an arbitrary Banach space. The following lemma and its corollaries are indispensable in this chapter.

Lemma 2.5.1. *Let $Q \subseteq \mathbf{R}^d$ be an open bounded cube, $p \ge 1$, $\delta \in (0,1]$, and f be a V-valued integrable function on Q such that*

$$[f]_{\delta;p;Q;V} := \left(\int_Q\int_Q \frac{|f(x)-f(y)|_V^p}{|x-y|^{2d+\delta p}}dxdy\right)^{1/p} < \infty.$$

Then f has a $C^\delta(Q;V)$-modification and there is a constant $N = N(d,\delta,p)$ independent of f and Q such that

$$[f]_{\delta;Q;V} \le N\,[f]_{\delta,p;Q;V}$$

and

$$\sup_{x\in Q}|f(x)|_V \le N|Q|^{\delta/d}[f]_{\delta;p;Q;V} + |Q|^{-1/p}\left(\int_Q |f(x)|_V^p dx\right)^{1/p},$$

where $|Q|$ is the volume of the cube.

Proof. If $V = \mathbf{R}$, then the existence of a continuous modification of f and the estimate of $[f]_{\delta;Q}$ follows from Lemma 2 and Exercise 5 in Chapter 10, Section 1, in [Kry08]. The proof for a general Banach space is the same. For all $x \in Q$, we have

$$\begin{aligned}
|f(x)|_V &\le \frac{1}{|Q|}\int_Q |f(x)-f(y)|_V dy + \frac{1}{|Q|}\int_Q |f(y)|_V dy \\
&\le N\frac{1}{|Q|}[f]_{\delta,p;Q}\int_Q |x-y|^\delta dy + \frac{1}{|Q|}\int_Q |f(y)|_V dy \\
&\le N|Q|^{\delta/d}[f]_{\delta,p;Q} + |Q|^{-1/p}\left(\int_Q |f(y)|_V^p dy\right)^{1/p},
\end{aligned}$$

which proves the second estimate. □

The following is a direct consequence of Lemma 2.5.1.

Corollary 2.5.2. *Let $p \ge 1$, $\delta \in (0,1]$, and f be a V-valued function on $\mathbf{R}^d$ such that*

$$|f|_{\delta;p;V} := \left(\int_{\mathbf{R}^d} |f(x)|_V^p dx + \int_{|x-y|<1} \frac{|f(x)-f(y)|_V^p}{|x-y|^{2d+\delta p}} dxdy\right)^{1/p} < \infty.$$

Then f has a $C^\delta(\mathbf{R}^d;V)$-modification and there is a constant $N = N(d,\delta,p)$ independent of f such that

$$|f|_{\delta;V} \le N|f|_{\delta;p;V}.$$

Corollary 2.5.3. *Let X be a V-valued random field defined on $\mathbf{R}^d$. Assume that for some $p \ge 1$, $l \ge 0$, and $\beta \in (0,1]$ with $\beta p > d$ there is a constant $\bar{N} > 0$ such that for all $x, y \in \mathbf{R}^d$,*

$$\mathbf{E}\left[|X(x)|_V^p\right] \le \bar{N} r_1(x)^{lp} \tag{2.5.1}$$

and

$$\mathbf{E}\left[|X(x)-X(y)|_V^p\right] \le \bar{N}[r_1(x)^{lp} + r_1(y)^{lp}]|x-y|^{\beta p}. \tag{2.5.2}$$

Then for all $\delta \in (0, \beta - \frac{d}{p})$ and $\epsilon > \frac{d}{p}$, there exists a $C^\delta(\mathbf{R}^d;V)$-modification of $r_1^{-(l+\epsilon)}X$ and a constant $N = N(d,p,\delta,\epsilon)$ such that

$$\mathbf{E}\left[|r_1^{-(l+\epsilon)}X|_\delta^p\right] \le N\bar{N}.$$

Proof. Fix $\delta \in (0, \beta - \frac{d}{p})$ and $\epsilon > \frac{d}{p}$. Owing to (2.5.1), there is a constant $N = N(d, p, \bar{N}$

$, \delta, \epsilon)$ such that

$$\int_{\mathbf{R}^d} \mathbf{E}\left[|r_1(x)^{-(l+\epsilon)} X(x)|_V^p\right] dx \leq \bar{N} \int_{\mathbf{R}^d} r_1(x)^{-p\epsilon} dx \leq N\bar{N}.$$

Appealing to (2.5.2) and (2.3.19), we find that there is a constant $N = N(d, p, \delta, \epsilon)$ such that

$$\begin{aligned}
&\int_{|x-y|<1} \frac{\mathbf{E}\left[|r_1(x)^{-(l+\epsilon)} X(x) - r_1(y)^{-(l+\epsilon)} X(y)|_V^p\right]}{|x-y|^{2d+\delta p}} dxdy \\
&\leq \bar{N} \int_{|x-y|<1} \frac{r_1(x)^{-p\epsilon} + r_1(y)^{-p\epsilon}}{|x-y|^{2d-(\beta-\delta)p}} dxdy + \bar{N} \int_{|x-y|<1} \frac{r_1(y)^{pl} |r_1(x)^{-(l+\epsilon)} - r_1(y)^{-(l+\epsilon)}|^p}{|x-y|^{2d+\delta p}} dxdy \\
&\leq N\bar{N} + N\bar{N} \int_{|x-y|<1} \frac{r_1(x)^{-p(1+\epsilon)} + r_1(y)^{-p(1+\epsilon)}}{|x-y|^{2d-(1-\delta)p}} dxdy \leq N\bar{N}.
\end{aligned}$$

Therefore, $\mathbf{E}[r_1^{-(l+\epsilon)} X]_{\delta,p}^p \leq N\bar{N}$, and hence, by Corollary 2.5.3, $r_1^{-(l+\epsilon)} X$ has a $C^\delta(\mathbf{R}^d; V)$-modification and the estimate follows immediately. □

Corollary 2.5.4. *Let $(X^{(n)})_{n\in\mathbf{N}}$ be a sequence of V-valued random fields defined on $\mathbf{R}^d$. Assume that for some $p \geq 1$, $l \geq 0$ and $\beta \in (0, 1]$, with $\beta p > d$ there is a constant $\bar{N} > 0$ such that for all $x, y \in \mathbf{R}^d$ and $n \in \mathbf{N}$,*

$$\mathbf{E}\left[|X^{(n)}(x)|_V^p\right] \leq \bar{N} r_1(x)^{lp}$$

and

$$\mathbf{E}\left[|X^{(n)}(x) - X^{(n)}(y)|_V^p\right] \leq \bar{N}(r_1(x)^{lp} + r_1(y)^{lp})|x-y|^{\beta p}.$$

Moreover, assume that for all $x \in \mathbf{R}^d$, $\lim_{n\to\infty} \mathbf{E}\left[|X^{(n)}(x)|^p\right] = 0$. Then for all $\delta \in (0, \beta - \frac{d}{p})$ and $\epsilon > \frac{d}{p}$,

$$\lim_{n\to\infty} \mathbf{E}\left[|r_1^{-(l+\epsilon)} X^{(n)}|_\delta^p\right] = 0.$$

Proof. Fix $\delta \in (0, \beta - \frac{d}{p})$ and $\epsilon > \frac{d}{p}$. Using the Lebesgue dominated convergence theorem, we attain

$$\lim_{n\to\infty} \int_{\mathbf{R}^d} \mathbf{E}\left[|r_1(x)^{-(l+\epsilon)} X^{(n)}(x)|_V^p\right] dx = 0,$$

and therefore for all $\zeta \in (0, 1)$,

$$\lim_n \int_{\zeta<|x-y|<1} \frac{\mathbf{E}\left[|r_1(x)^{-(l+\epsilon)} X_n(x) - r_1(y)^{-(l+\epsilon)} X_n(y)|_V^p\right]}{|x-y|^{2d+\delta p}} dxdy = 0.$$

By repeating the proof of Corollary 2.5.3, we obtain that there is a constant N such that

$$\begin{aligned}
&\int_{|x-y|\le\zeta} \frac{\mathbf{E}\left[|r_1(x)^{-(l+\epsilon)}X^{(n)}(x) - r_1(y)^{-(l+\epsilon)}X^{(n)}(y)|_V^p\right]}{|x-y|^{2d+\delta p}}dxdy \\
&\le \bar{N}\int_{|x-y|\le\zeta} \frac{r_1(x)^{-p\epsilon} + r_1(y)^{-p\epsilon}}{|x-y|^{2d+(\delta-\beta)p}}dxdy + \bar{N}\int_{|x-y|\le\zeta} \frac{r_1(x)^{-p(1+\epsilon)} + r_1(y)^{-p(1+\epsilon)}}{|x-y|^{2d+(\delta-1)p}}dxdy \\
&\le \bar{N}\zeta^{\beta p-\delta p-d}.
\end{aligned}$$

Therefore, $\lim_{n\to\infty}\mathbf{E}\left[[r_1^{-(l+\epsilon)}X]_{\delta,p}^p\right] = 0$, and the statement is confirmed. □

Chapter 3

The method of stochastic characteristics for parabolic SIDEs

3.1 Introduction

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space and $\tilde{\mathcal{F}}_0$ be a sub-sigma-algebra of $\mathcal{F}$. We assume that this probability space supports a sequence $w_t^{1;\varrho}$, $t \geq 0$, $\varrho \in \mathbf{N}$, of independent one-dimensional Wiener processes and a Poisson random measure $p^1(dt, dz)$ on $(\mathbf{R}_+ \times Z^1, \mathcal{B}(\mathbf{R}_+) \otimes \mathcal{Z}^1)$ with intensity measure $\pi^1(dz)dt$, where $(Z^1, \mathcal{Z}^1, \pi^1)$ is a sigma-finite measure space. We also assume that $(w_t^{1;\varrho})_{\varrho \in \mathbf{N}}$ and $p^1(dt, dz)$ are independent of $\mathcal{F}_0$. Let $\mathbf{F} = (\mathcal{F}_t)_{t \geq 0}$ be the standard augmentation of the filtration $(\bar{\mathcal{F}}_t)_{t \geq 0}$, where for each $t \geq 0$,

$$\bar{\mathcal{F}}_t = \sigma\left(\tilde{\mathcal{F}}_0, (w_s^{1;\varrho})_{\varrho \in \mathbf{N}},\ p^1([0, s], \Gamma) : s \leq t,\ \Gamma \in \mathcal{Z}^1\right).$$

For each real number $T > 0$, we let $\mathcal{R}_T$, $\mathcal{O}_T$, and $\mathcal{P}_T$ be the $\mathbf{F}$-progressive, $\mathbf{F}$-optional, and $\mathbf{F}$-predictable sigma-algebra on $\Omega \times [0, T]$, respectively. Denote by $q^1(dt, dz) = p^1(dt, dz) - \pi^1(dz)dt$ the compensated Poisson random measure. Let $D^1, E^1, V^1 \in Z$ be disjoint $\mathcal{Z}^1$-measurable subsets such that $D^1 \cup E^1 \cup V^1 = Z^1$ and $\pi(V^1) < \infty$. Let $(Z^2, \mathcal{Z}^2, \pi^2)$ be a sigma-finite measure space and $D^2, E^2 \in Z^2$ be disjoint $\mathcal{Z}^2$-measurable subsets such that $D^2 \cup E^2 = Z^2$.

Fix an arbitrary positive real number $T > 0$ and integers $d_1, d_2 \geq 1$. Let $\alpha \in (0, 2]$ and $\tau \leq T$ be a stopping time. Let $\mathcal{F}_\tau$ be the stopping time sigma-algebra associated with τ and let $\varphi : \Omega \times \mathbf{R}^{d_1} \to \mathbf{R}^{d_2}$ be $\mathcal{F}_\tau \otimes \mathcal{B}(\mathbf{R}^{d_1})$-measurable. We consider the system of SIDEs on $[0, T] \times \mathbf{R}^{d_1}$ given by

$$\begin{aligned} du_t^l &= \left((\mathcal{L}_t^{1;l} + \mathcal{L}_t^{2;l})u_t + \mathbf{1}_{[1,2]}(\alpha)b_t^i\partial_i u_t^l + c_t^{l\bar{l}}u_t^{\bar{l}} + f_t^l\right)dt + \left(\mathcal{N}_t^{1;l\varrho}u_t + g_t^{l\varrho}\right)dw_t^{1;\varrho} \\ &\quad + \int_{Z^1}\left(\mathcal{I}_{t,z}^{1;l}u_{t-} + h_t^l(z)\right)[\mathbf{1}_{D^1}(z)q^1(dt, dz) + \mathbf{1}_{E^1 \cup V^1}(z)p^1(dt, dz)], \quad \tau \leq t \leq T, \\ u_t^l &= \varphi^l, \quad t \leq \tau, \quad l \in \{1, \ldots, d_2\}, \end{aligned} \tag{3.1.1}$$

where for $\phi \in C_c^\infty(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$, $k \in \{1, 2\}$, and $l \in \{1, \ldots, d_2\}$,

$$\mathcal{L}_t^{k;l}\phi(x) := \mathbf{1}_{\{2\}}(\alpha)\frac{1}{2}\sigma_t^{k;i\varrho}(x)\sigma_t^{k;j\varrho}(x)\partial_{ij}\phi^l(x) + \mathbf{1}_{\{2\}}(\alpha)\sigma_t^{k;i\varrho}(x)\upsilon_t^{k;l\bar{l}\varrho}(x)\partial_i\phi^{\bar{l}}(x)$$

$$
\begin{aligned}
&+\int_{D^k}\rho_t^{k;l\bar l}(x,z)\left(\phi^{\bar l}(x+H_t^k(x,z))-\phi^{\bar l}(x)\right)\pi^k(dz)\\
&+\int_{D^k}\left(\phi^{l}(x+H_t^k(x,z))-\phi^{l}(x)-\mathbf{1}_{(1,2]}(\alpha)H_t^{k;i}(x,z)\partial_i\phi^l(x)\right)\pi^k(dz)\\
&+\mathbf{1}_{\{2\}}(k)\int_{E^2}\left((I_{d_2}^{l\bar l}+\rho_t^{2;l\bar l}(x,z))\phi^{\bar l}(x+H_t^2(x,z))-\phi^l(x)\right)\pi^2(dz),\\
\mathcal{N}_t^{1;l\varrho}\phi(x) &:= \mathbf{1}_{\{2\}}(\alpha)\sigma_t^{1;i\varrho}(x)\partial_i\phi^l(x)+\upsilon_t^{1;l\bar l\varrho}(x)\phi^{\bar l}(x),\quad \varrho\in\mathbf{N},\\
\mathcal{I}_{t,z}^{1;l}\phi(x) &:= (I_{d_2}^{l\bar l}+\rho_t^{1;l\bar l}(x,z))\phi^{\bar l}(x+H_t^1(x,z))-\phi^l(x),
\end{aligned}
$$

and

$$
\int_{D^k}\left(|H_t^k(x,z)|^{\alpha}+|\rho_t^k(x,z)|^2\right)\pi^k(dz)+\int_{E^k}\left(|H_t^k(x,z)|^{1\wedge\alpha}+|\rho_t^k(x,z)|\right)\pi^k(dz)<\infty.
$$

The summation convention with respect to repeated indices $i,j\in\{1,\dots,d_1\}$,$\bar l\in\{1,\dots,d_2\}$, and $\varrho\in\mathbf{N}$ is used here and below. The $d_2\times d_2$ dimensional identity matrix is denoted by I_{d_2}. For a subset A of a larger set X, $\mathbf{1}_A$ denotes the $\{0,1\}$-valued function taking the value 1 on the set A and 0 on the complement of A. We assume that for each $k\in\{1,2\}$,

$$
\sigma_t^k(x)=(\sigma_t^{k;i\varrho}(\omega,x))_{1\le i\le d_1,\,\varrho\in\mathbf{N}},\quad b_t(x)=(b_t^i(\omega,x))_{1\le i\le d_1},\quad c_t(x)=(c_t^{l\bar l}(\omega,x))_{1\le l,\bar l\le d_2},
$$

$$
\upsilon_t^k(x)=(\upsilon_t^{k;l\bar l\varrho}(\omega,x))_{1\le l,\bar l\le d_2,\,\varrho\in\mathbf{N}},\quad f_t(x)=(f_t^i(\omega,x))_{1\le i\le d_2},\quad g_t(x)=(g_t^{i\varrho}(\omega,x))_{1\le i\le d_2,\,\varrho\in\mathbf{N}},
$$

are random fields on $\Omega\times[0,T]\times\mathbf{R}^{d_1}$ that are $\mathcal{R}_T\otimes\mathcal{B}(\mathbf{R}^{d_1})$-measurable. For each $k\in\{1,2\}$, we assume that

$$
H_t^k(x,z)=(H_t^{k;i}(\omega,x,z))_{1\le i\le d_1},\ \rho_t^k(x,z)=(\rho_t^{k;l\bar l}(\omega,x,z))_{1\le l,\bar l\le d_2},
$$

are random fields on $\Omega\times[0,T]\times\mathbf{R}^{d_1}\times Z^k$ that are $\mathcal{P}_T\otimes\mathcal{B}(\mathbf{R}^{d_1})\otimes\mathcal{Z}^k$-measurable. Moreover, we assume that $h_t(x,z)=(h_t^i(\omega,x,z))_{1\le i\le d_2}$ is a random field on $\Omega\times[0,T]\times\mathbf{R}^{d_1}$ that is $\mathcal{P}_T\otimes\mathcal{B}(\mathbf{R}^{d_1})$-measurable. Systems of linear SIDEs appear in many contexts. They may be considered as extensions of both first-order symmetric hyperbolic systems and linear fractional advection-diffusion equations. The equation (3.1.1) also arises in non-linear filtering of semimartingales as the equation for the unormalized filter of the signal (see, e.g., [Gri76] and [GM11]). Moreover, (3.1.1) is intimately related to linear transformations of inverse flows of jump SDEs and it is precisely this connection that we will exploit to obtain solutions.

Systems of linear SIDEs appear in many contexts. They may be considered as extensions of both first-order symmetric hyperbolic systems and linear fractional advection-diffusion equations. The equation (3.1.1) also arises in non-linear filtering of semimartingales as the equation for the unormalized filter of the signal (see, e.g., [Gri76] and

[GM11]). Moreover, (3.1.1) is intimately related to linear transformations of inverse flows of jump SDEs and it is precisely this connection that we will exploit to obtain solutions.

There are various techniques available to derive the existence and uniqueness of classical solutions of linear parabolic SPDEs and SIDEs. One approach is to develop a theory of weak solutions for the equations (e.g. variational, mild solution, or etc...) and then study further regularity in classical function spaces via an embedding theorem. We refer the reader to [Par72, Par75, MP76, KR77, Tin77a, Gyö82, Wal86, DPZ92, Kry99, CK10, PZ07, Hau05, RZ07, BvNVW08, HØUZ10, LM14b] for more information about weak solutions of SPDEs driven by continuous and discontinuous martingales and martingale measures. This approach is especially important in the non-degenerate setting where some smoothing occurs and has the obvious advantage that it is broader in scope. Another approach is to regard the solution as a function with values in a probability space and use the method of deterministic PDEs (i.e. Schauder estimates, see, e.g. [Mik00, MP09]). A third approach is a direct one that uses solutions of stochastic differential equations. The direct method allows to obtain classical solutions in the entire Hölder scale while not restricting to integer derivative assumptions for the coefficients and data. In this chapter, we derive the existence of a classical solutions of (3.1.1) with regular coefficients using a Feynman-Kac-type transformation and the interlacing of the space-inverse (first integrals [KR81]) of a stochastic flow associated with the equation. The construction of the solution gives an insight into the structure of the solution as well. We prove that the solution of (3.1.1) is unique in the class of classical solutions with polynomial growth (i.e. weighted Hölder spaces). As an immediate corollary of our main result, we obtain the existence and uniqueness of classical solutions of linear partial integro-differential equations (PIDEs) with random coefficients, since the coefficients σ^1, H^1, a^1, ρ^1, and free terms g and h can be zero. Our work here directly extends the method of characteristics for deterministic first-order PDEs and the well-known Feynman-Kac formula for deterministic second-order PDEs.

In the continuous case (i.e. $H^1 \equiv 0, H^2 \equiv 0, h \equiv 0$), the classical solution of (3.1.1) was constructed in [KR81, Kun81, Kun86a, Roz90] (see references therein as well) using the first integrals of the associated backward SDE. This method was also used to obtain classical solutions of (3.1.1) in [DPMT07]. In the references above, the forward Liouville equation for the first integrals of associated stochastic flow was derived directly. However, since the backward equation involves a time reversal, the coefficients and input functions are assumed to be non-random. The generalized solutions of (3.1.1) with $d_2 = 1$, non-random coefficients, non-degenerate diffusion, and finite measures $\pi^1 = \pi^2$ were discussed in [MB07]. In this chapter, we give a direct derivation of (3.1.1) and all the equations considered are forward, possibly degenerate, and the coefficients and input functions are adapted. For other interesting and related developments, we refer the reader to [Pri12, Zha13, Pri14], which all concern the fascinating regularizing property of noise in the case of non-degenerate noise.

This chapter is organized as follows. In Section 3.2, our notation is set forth and the main results are stated. In Section 3.3, the main theorems are proved. In Section 3.4, the appendix, auxiliary facts used throughout the chapter are discussed.

3.2 Statement of main results

In this chapter, elements of $\mathbf{R}^{d_1}$ and $\mathbf{R}^{d_2}$ are understood as column vectors and elements of $\mathbf{R}^{d_1^2}$ and $\mathbf{R}^{d_2^2}$ are understood as matrices of dimension $d_1 \times d_1$ and $d_2 \times d_2$, respectively. We also adopt the notation of Chapter 2. If we do not specify to which space the parameters ω, t, x, y, z and n belong, then we mean $\omega \in \Omega$, $t \in [0,T]$, $x, y \in \mathbf{R}^{d_1}$, $z \in Z^k$, and $n \in \mathbf{N}$.

Let us introduce some regularity conditions on the coefficients and free terms. We consider these assumptions for $\bar{\beta} > 1 \vee \alpha$ and $\tilde{\beta} > \alpha$.

Assumption 3.2.1 ($\bar{\beta}$). *(1) There is a constant $N_0 > 0$ such that for each $k \in \{1,2\}$ and all $\omega, t \in \Omega \times [0,T]$,*

$$|r_1^{-1} b_t|_0 + |\nabla b_t|_{\bar{\beta}-1} + |r_1^{-1}\sigma_t^k|_0 + |\nabla \sigma_t^k|_{\bar{\beta}-1} \le N_0.$$

Moreover, for each $k \in \{1,2\}$ and all $(\omega,t,z) \in \Omega \times [0,T] \times (D^k \cup E^k)$,

$$|r_1^{-1} H_t^k(z)|_0 \le K_t^k(z) \quad \text{and} \quad |\nabla H_t^k(z)|_{\bar{\beta}-1} \le \bar{K}_t^k(z)$$

where $K^k, \bar{K}^k : \Omega \times [0,T] \times (D^k \cup E^k) \to \mathbf{R}_+$ are $\mathcal{P}_T \otimes \mathcal{Z}^k$-measurable functions satisfying

$$K_t^k(z) + \bar{K}_t^k(z) + \int_{D^k} \left(K_t^k(z)^\alpha + \bar{K}_t^k(z)^2\right)\pi^k(dz) + \int_{E^k} \left(K_t^k(z)^{1\wedge\alpha} + \bar{K}_t^k(z)\right)\pi^k(dz) \le N_0,$$

for all $(\omega,t,z) \in \Omega \times [0,T] \times (D^k \cup E^k)$.

(2) There is a constant $\eta \in (0,1)$ such that for all $(\omega,t,x,z) \in \{(\omega,t,x,z) \in \Omega \times [0,T] \times \mathbf{R}^{d_1} \times (D^k \cup E^k) : |\nabla H_t^k(\omega,x,z)| > \eta\}$,

$$\left|\left(I_{d_1} + \nabla H_t^k(x,z)\right)^{-1}\right| \le N_0.$$

Assumption 3.2.2 ($\tilde{\beta}$). *There is a constant $N_0 > 0$ such that for each $k \in \{1,2\}$ and all $(\omega,t) \in \Omega \times [0,T]$,*

$$|c_t|_{\tilde{\beta}} + |\upsilon_t^k|_{\tilde{\beta}} + |r_1^{-\theta} f_t|_{\tilde{\beta}} + |r_1^{-\theta} g_t|_{\tilde{\beta}} \le N_0.$$

Moreover, for each $k \in \{1,2\}$ and all $(\omega,t,z) \in \Omega \times [0,T] \times (D^k \cup E^k)$,

$$|\rho_t^k(z)|_{\tilde{\beta}} \leq l_t^k(z), \quad |r_1^{-\theta} h_t(z)|_{\tilde{\beta}} \leq l_t^k(z),$$

where $l^k : \Omega \times [0,T] \times Z^k \to \mathbf{R}_+$ are $\mathcal{P}_T \otimes \mathcal{Z}^k$-measurable function satisfying

$$l_t^k(z) + \int_{D^k} l_t^k(z)^2 \pi^k(dz) + \int_{E^k} l_t^k(z) \pi^k(dz) \leq N_0,$$

for all $(\omega,t,z) \in \Omega \times [0,T] \times (D^k \cup E^k)$.

Remark 3.2.1. It follows from Lemma 3.4.10 and Remark 3.4.11 that if Assumption 3.2.1($\bar{\beta}$) holds for some $\bar{\beta} > 1 \vee \alpha$, then for all ω, t, and $z \in D^k \cup E^k$, $x \mapsto \tilde{H}_t^k(x,z) := x + H_t^k(x,z)$ is a diffeomorphism.

Let Assumptions 3.2.1($\bar{\beta}$) and 3.2.2($\tilde{\beta}$) hold for some $\bar{\beta} > 1 \vee \alpha$ and $\tilde{\beta} > \alpha$. In our derivation of a solution of (3.1.1), we first obtain a solution of an equation of a special form. Specifically, consider the system of SIDEs on $[0,T] \times \mathbf{R}^{d_1}$ given by

$$\begin{aligned} d\hat{u}_t^l &= \left((\mathcal{L}_t^{1;l} + \mathcal{L}_t^{2;l})\hat{u}_t + \hat{b}_t^i \partial_i u_t^l + \hat{c}_t^{l\bar{l}} u_t^{\bar{l}} + \hat{f}_t^l \right) dt + \left(\mathcal{N}_t^{1;l\varrho} \hat{u}_t + g_t^{l\varrho} \right) dw_t^{1;\varrho} \\ &\quad + \int_{Z^1} \left(\mathcal{I}_{t,z}^{1;l} \hat{u}_{t-} + h_t^l(z) \right) [\mathbf{1}_{D^1}(z) q^1(dt,dz) + \mathbf{1}_{E^1}(z) p^1(dt,dz)], \quad \tau < t \leq T, \\ \hat{u}_t^l &= \varphi^l, \quad t \leq \tau, \quad l \in \{1,\dots,d_2\}, \end{aligned} \tag{3.2.1}$$

where

$$\begin{aligned} \hat{b}_t^i(x) :&= \mathbf{1}_{[1,2]}(\alpha) b_t^i(x) + \sum_{k=1}^2 \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{k;j\varrho}(x) \partial_j \sigma_t^{k;i\varrho}(x) \\ &\quad + \sum_{k=1}^2 \mathbf{1}_{(1,2]}(\alpha) \int_{D^k} \left(H_t^{k;i}(x,z) - H_t^{k;i}(\tilde{H}_t^{k;-1}(x,z),z) \right) \pi^k(dz), \\ \hat{c}_t^{l\bar{l}}(x) :&= c_t^{l\bar{l}}(x) + \sum_{k=1}^2 \mathbf{1}_{\{2\}}(\alpha) \sigma_t^{k;j\varrho}(x) \partial_j \upsilon_t^{k;l\bar{l}\varrho}(x) \\ &\quad + \sum_{k=1}^2 \int_{D^k} (\rho_t^{k;l\bar{l}}(x,z) - \rho_t^{k;l\bar{l}}(\tilde{H}_t^{k;-1}(x,z),z)) \pi^k(dz), \\ \hat{f}_t^l(x) :&= f_t^l(x) + \sigma_t^{1;j\varrho}(x) \partial_j g_t^{l\varrho}(x) + \int_{D^1} \left(h_t^l(x,z) - h_t^l(\tilde{H}_t^{1;-1}(x,z),z) \right) \pi^1(dz). \end{aligned}$$

Let $(w_t^{2;\varrho})_{\varrho \in \mathbf{N}}$, $t \geq 0$, $\varrho \in \mathbf{N}$, be a sequence of independent one-dimensional Wiener processes. Let $p^2(dt,dz)$ be a Poisson random measure on $([0,\infty) \times Z^2, \mathcal{B}([0,\infty) \otimes \mathcal{Z}^2)$ with intensity measure $\pi^2(dz)dt$. Extending the probability space if necessary, we take w^2

and $p^2(dt,dz)$ to be independent of w^1 and $p^1(dt,dz)$. Let

$$\hat{\mathcal{F}}_t = \sigma\left((w_s^{2;\varrho})_{\varrho\in\mathbf{N}},\ p^2([0,s],\Gamma) : s \le t,\ \Gamma \in \mathcal{Z}^2\right)$$

and $\tilde{\mathbf{F}} = \left(\tilde{\mathcal{F}}_t\right)_{t\le T}$ be the standard augmentation of $\left(\mathcal{F}_t \vee \hat{\mathcal{F}}_t\right)_{t\le T}$. Denote by $q^2(dt,dz) = p^2(dt,dz) - \pi^2(dz)dt$ the compensated Poisson random measure. We associate with the SIDE (3.2.1), the $\tilde{\mathbf{F}}$-adapted stochastic flow $X_t = X_t(x) = X_t(\tau,x)$, $(t,x)\in[0,T]\times\mathbf{R}^{d_1}$, generated by the SDE

$$\begin{aligned}
dX_t &= -\mathbf{1}_{[1,2]}(\alpha)b_t(X_t)dt + \sum_{k=1}^{2}\mathbf{1}_{\{2\}}(\alpha)\sigma_t^{k;\varrho}(X_t)dw_t^{k;\varrho}\\
&\quad - \sum_{k=1}^{2}\int_{D^k} H_t^k(\tilde{H}_t^{k;-1}(X_{t-},z),z)[p^k(dt,dz) - \mathbf{1}_{(1,2]}(\alpha)\pi^k(dz)dt]\\
&\quad - \sum_{k=1}^{2}\int_{E^k} H_t^k(\tilde{H}_t^{k;-1}(X_{t-},z),z)p^k(dt,dz), \quad \tau < t \le T,\\
X_t &= x, \quad t\le\tau,
\end{aligned} \tag{3.2.2}$$

and the $\tilde{\mathbf{F}}$-adapted random field $\Phi_t(x) = \Phi_t(\tau,x)$, $(t,x)\in[0,T]\times\mathbf{R}^{d_1}$, solving the linear SDE given by

$$\begin{aligned}
d\Phi_t(x) &= (c_t(X_t(x))\Phi_t(x) + f_t(X_t(x)))\,dt + \sum_{k=1}^{2}\upsilon_t^{k;\varrho}(X_t(x))\Phi_t(x)dw_t^{k;\varrho} + g_t^{\varrho}(X_t(x))dw_t^{1;\varrho}\\
&\quad + \sum_{k=1}^{2}\int_{Z^k}\rho_t^k(\tilde{H}_t^{k;-1}(X_{t-}(x),z),z)\Phi_{t-}(x)[\mathbf{1}_{D^k}(z)q^k(dt,dz) + \mathbf{1}_{E^k}(z)p^k(dt,dz)]\\
&\quad + \int_{Z^1} h_t(\tilde{H}_t^{1;-1}(X_{t-}(x),z),z)[\mathbf{1}_{D^1}(z)q^1(dt,dz) + \mathbf{1}_{E^1}(z)p^1(dt,dz)], \quad \tau<t\le T,\\
\Phi_t(x) &= \varphi(x), \quad t\le\tau.
\end{aligned}$$

The coming theorem is our existence, uniqueness, and representation theorem for (3.2.1). Let us describe our solution class. For each $\beta'\in(0,\infty)$, denote by $\mathfrak{C}^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$ the linear space of all $\mathbf{F}$-adapted random fields $v = v_t(x)$ such that $\mathbf{P}$-a.s.

$$\mathbf{1}_{[\tau_n,\tau_{n+1})}r_1^{-\lambda_n}v \in D([0,T];C^{\beta'}(\mathbf{R}^{d_1},\mathbf{R}^{d_2})),$$

where $(\tau_n)_{n\ge0}$ is an increasing sequence of $\mathbf{F}$-stopping times with $\tau_0 = 0$ and $\tau_n = T$ for sufficiently large n, and where for each n, λ_n is a positive $\mathcal{F}_{\tau_n}$-measurable random variable.

Theorem 3.2.2. *Let Assumptions 3.2.1($\bar\beta$) and 3.2.2($\tilde\beta$) hold for some $\bar\beta > 1\vee\alpha$ and $\tilde\beta>\alpha$. For any stopping time $\tau\le T$ and $\mathcal{F}_\tau\otimes\mathcal{B}(\mathbf{R}^{d_1})$-measurable random field φ such that for* *some* $\beta' \in (\alpha, \bar\beta \wedge \tilde\beta)$ *and* $\theta' \geq 0$, $\mathbf{P}$*-a.s.* $r_1^{-\theta'}\varphi \in C^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$*, there exists a unique solution* $\hat u = \hat u(\tau)$ *of* (3.2.1) *in* $\mathfrak{C}^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$ *and for all* $(t,x) \in [0,T]\times\mathbf{R}^{d_1}$, $\mathbf{P}$*-a.s.*

$$\hat u_t(\tau,x) = \mathbf{E}\left[\Phi_t(\tau, X_t^{-1}(\tau,x))|\mathcal{F}_t\right]. \tag{3.2.3}$$

Moreover, for all $\epsilon > 0$ *and* $p \geq 2$*, there is a constant* $N = N(d_1,d_2,p,N_0,T,\beta',\eta^1,\eta^2,\epsilon,\theta,\theta')$ *such that*

$$\mathbf{E}\left[\sup_{t\leq T}|r_1^{-\theta\vee\theta'-\epsilon}\hat u_t(\tau)|^p_{\beta'}\Big|\mathcal{F}_\tau\right] \leq N(|r_1^{-\theta'}\varphi|^p_{\beta'}+1). \tag{3.2.4}$$

Using Itô's formula, it is easy to verify that if $m = 1$ and

$$g_t(x) = 0, \quad h_t(x) = 0, \quad \text{and} \quad \rho_t^k(x,z) \geq -1,$$

for all $(\omega,t,x,z) \in \Omega\times[0,T]\times\mathbf{R}^{d_1}\times(D^k\cup E^k)$, $k \in \{1,2\}$, then

$$\Phi_t(x) = \Psi_t(x)\phi(x) + \Psi_t(x)\int_{]\tau,\tau\vee t]}\Psi_s^{-1}(x)f_s(X_s(x))ds, \tag{3.2.5}$$

where $\mathbf{P}$-a.s. for all t and x, $\Psi_t(x) := e^{\mathcal{B}_t(x)}$, and

$$\begin{aligned}\mathcal{B}_t(x) := &\int_{[\tau,\tau\vee t]}\left(c_s(X_s(x)) - \sum_{k=1}^2\frac{1}{2}\upsilon_s^{k;\varrho}(X_s(x))\upsilon_s^{k;\varrho}(X_s(x))\right)ds\\ &+\sum_{k=1}^2\int_{]\tau,\tau\vee t]}\upsilon_s^{k;\varrho}(X_s(x))dw_s^{k;\varrho}\\ -\sum_{k=1}^2\int_{]\tau,\tau\vee t]}&\int_{D^k}\left(\ln\left(1+\rho_s^k(\tilde H_s^{k;-1}(X_{s-}(x),z),z)\right) - \rho_s^k(\tilde H_s^{k;-1}(X_{s-}(x),z),z)\right)\pi^k(dz)ds\\ \sum_{k=1}^2\int_{]\tau,\tau\vee t]}&\int_{Z^k}\ln\left(1+\rho_s^k(\tilde H_s^{k;-1}(X_{s-}(x),z),z)\right)[\mathbf{1}_{D^k}(z)q^k(ds,dz)+\mathbf{1}_{E^k}(z)p^k(ds,dz)].\end{aligned}$$

The following corollary then follows directly from (3.2.3) and (3.2.5).

Corollary 3.2.3. *Let* $m = 1$ *and assume that for each* $k \in \{1,2\}$ *and all* $(\omega,t,x,z) \in \Omega\times[0,T]\times\mathbf{R}^{d_1}\times(D^k\cup E^k)$,

$$g_t(x) = 0, \quad h_t(x,z) = 0, \quad \rho_t^k(x,z) \geq -1.$$

Moreover, let Assumptions 3.2.1$(\bar\beta)$ *and* 3.2.2$(\tilde\beta)$ *hold for some* $\bar\beta > 1\vee\alpha$ *and* $\tilde\beta > \alpha$*. Let* $\tau \leq T$ *be a stopping time and* φ *be a* $\mathcal{F}_\tau\otimes\mathcal{B}(\mathbf{R}^{d_1})$*-measurable random field such that for some* $\beta' \in (\alpha,\bar\beta\wedge\tilde\beta)$ *and* $\theta' \geq 0$, $\mathbf{P}$*-a.s.* $r_1^{-\theta'}\varphi \in C^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$.

(1) If for all $(\omega,t,x) \in \Omega\times[0,T]\times\mathbf{R}^{d_1}$, $f_t(x) \geq 0$ *and* $\varphi(x) \geq 0$*, then the solution* $\hat u$ *of* (3.1.1) *satisfies* $\hat u_t(x) \geq 0$, $\mathbf{P}$*-a.s. for all* $(t,x) \in [0,T]\times\mathbf{R}^{d_1}$.

(2) *If for all* $(\omega,t,x,z)\in\Omega\times[0,T]\times\mathbf{R}^{d_1}\times(D^k\cup E^k)$, $k\in\{1,2\}$, $\upsilon^k_t(x)=0$, $f_t(x)\le 0$, $c_t(x)\le 0$, $\varphi(x)\le 1$, *and* $\rho^k_t(x,z)\le 0$, *then the solution* $\hat u$ *of (3.1.1) satisfies* $\hat u_t(x)\le 1$, $\mathbf{P}$*-a.s. for all* $(t,x)\in[0,T]\times\mathbf{R}^{d_1}$.

Remark 3.2.4. Since $\mathcal{L}^2$ can be the zero operator, both Theorem 3.2.2 and Corollary 3.2.3 apply to fully degenerate SIDEs and PIDEs with random coefficients.

Now, let us discuss our existence and uniqueness theorem for (3.1.1). We construct the solution of $u=u(\tau)$ of (3.1.1) by interlacing the solutions of (3.2.1) along a sequence of large jump moments (see Section 3.3.5). By using an interlacing procedure we are also able to drop the condition of boundedness of $(I+\nabla H^1_t(x,z))^{-1}$ on the set $(\omega,t,x,z)\in\{(\omega,t,x,z)\in\Omega\times[0,T]\times\mathbf{R}^{d_1}\times(D^1\cup E^1):|\nabla H^1_t(\omega,x,z)|>\eta^k\}$. Also, in order to remove the terms in $\hat b$, $\hat c$, and $\hat f$ that appear in (3.2.1), but not in (3.1.1), we subtract terms from the relevant coefficients in the flow and the transformation. However, in order to do this, we need to impose stronger regularity assumptions on some of the coefficients and free terms. We will introduce parameters $\mu^1,\mu^2,\delta^1,\delta^2\in[0,\frac{\alpha}{2}]$ to the regularity assumptions. These parameters allows for a trade-off of integrability in z and regularity in x of the coefficients $H^k_t(x,z),\rho^k_t(x,z),h^k_t(x,z)$. It is worth mentioning that the removal of terms and the interlacing procedure are independent of each other and that it is due only to the weak assumptions on H^1 and ρ^1 on the set V^1 that we do not have moment estimates and a simple representation property like (3.2.4) for the solution of (3.1.1). However, there is a representation of the solution and we refer the reader to the proof of the coming theorem for the description of the representation.

We introduce the following assumptions for $\bar\beta>1\vee\alpha$ and $\tilde\beta>\alpha$. For each $k\in\{1,2\}$, let $\bar{\mathcal{D}}^k$ be the trace sigma-algebra of $D^k\cup E^k$ relative to $\mathcal{Z}^k$ and $\mathcal{V}^1$ be the trace sigma-algebra of V^1 relative to $\mathcal{Z}^1$.

Assumption 3.2.3 ($\bar\beta$)**.** *(1) There is a constant* $N_0>0$ *such that for each* $k\in\{1,2\}$ *and all* $(\omega,t)\in\Omega\times[0,T]$,

$$|r_1^{-1}b_t|_0+|\nabla b_t|_{\bar\beta-1}+|\sigma^k_t|_{\bar\beta+1}\le N_0.$$

(2) For each $k\in\{1,2\}$ *, there are real-numbers* $\delta^k,\mu^k\in[0,\frac{\alpha}{2}]$ *such that for all* $(\omega,t)\in\Omega\times[0,T]$,

$$\begin{gathered}|\upsilon^k_t|_{\bar\beta+1}\le N_0,\ \text{if}\ \sigma^k_t\neq 0,\quad |g_t|_{\bar\beta+1}\le N_0,\ \text{if}\ \sigma^1_t\neq 0,\\ |H^k_t(z)|_0\le K^k_t(z),\quad |\nabla H^k_t(z)|_{\bar\beta+\delta^k-1}\le\bar K^k_t(z),\ \forall z\in D^k,\\ \sum_{|\gamma|=[\bar\beta+\delta^k]^-}[\partial^\gamma H^k_t(z)]_{\{\bar\beta+\delta^k\}^+}\le\tilde K^k_t(z),\ \forall z\in D^k,\\ |r_1^{-1}H^k_t(z)|_0\le K^k_t(z),\quad |\nabla H^k_t(z)|_{\bar\beta-1}\le\bar K^k_t(z),\ \forall z\in E^k,\end{gathered}$$

$$|\rho_t^k(z)|_{\bar\beta+\mu^k} \le l_t^k(z), \qquad \sum_{|\gamma|=[\bar\beta+\mu^k]^-} [\partial^\gamma \rho_t^k(z)]_{\{\bar\beta+\mu^k\}^+} \le \tilde l_t^k(z), \ \forall z \in D^k,$$
$$|r_1^{-\theta} h_t(z)|_{\bar\beta+\mu^1} \le l_t^1(z), \qquad \sum_{|\gamma|=[\bar\beta+\mu^1]^-} [\partial^\gamma h_t(z)]_{\{\bar\beta+\mu^1\}^+}, \ \forall z \in D^1,$$

where $K^k, \bar K^k, \tilde K^k, l^k, \tilde l^k : \Omega\times[0,T]\times(D^k\cup E^k) \to \mathbf{R}_+$ *are* $\mathcal{P}_T\otimes\bar{\mathcal{D}}^k$*-measurable functions satisfying for all* $(\omega,t,z) \in \Omega\times[0,T]$,

$$K_t^k(z) + \bar K_t^k(z) + \tilde l_t^k(z) + l_t^k(z) + \tilde l_t^k(z) \le N_0, \quad \forall z \in D^k\cup E^k,$$
$$\int_{D^k} \left(K_t^k(z)^\alpha + \bar K_t^k(z)^2 + \tilde K_t^k(z)^{\frac{\alpha}{\alpha-\delta^k}} + \bar K_t^k(z)^2 + l_t^k(z)^2 + \tilde l_t^k(z)^{\frac{\alpha}{\alpha-\mu^k}} + \tilde l_t^k(z)^2\right)\pi^k \le N_0(dz),$$
$$\int_{E^k} \left(K_t^k(z)^{1\wedge\alpha} + \bar K_t^k(z)\right)\pi^k(dz) \le N_0.$$

(3) *There is a constant* $\eta \in (0,1)$ *such that for all* $(\omega,t,x,z) \in \{(\omega,t,x,z) \in \Omega\times[0,T]\times \mathbf{R}^{d_1}\times Z^2 : |\nabla H_t^2(\omega,x,z)| > \eta\}$,

$$\left|\left(I_{d_1} + \nabla H_t^2(x,z)\right)^{-1}\right| \le N_0.$$

Assumption 3.2.4 ($\tilde\beta$). *(1) There is a constant* $N_0 > 0$ *such that for each* $k \in \{1,2\}$ *and all* $(\omega,t) \in \Omega\times[0,T]$,

$$|c_t|_{\tilde\beta} + |r_1^{-\theta} f_t|_{\tilde\beta} \le N_0,$$
$$|\upsilon_t^k|_{\tilde\beta} \le N_0, \ \text{if } \sigma_t^k = 0, \quad |g_t|_{\tilde\beta} \le N_0, \ \text{if } \sigma_t^1 = 0,$$
$$|\rho^k(t,z)|_{\tilde\beta} \le l_t^k(z), \ \ \forall z \in E^k, \quad |r_1^{-\theta} h_t(z)|_{\tilde\beta} \le l_t^1(z), \ \forall z \in E^1,$$

where for all $(\omega,t) \in \Omega\times[0,T]$, $\int_{E^k} l_t^k(z)\pi^k(dz) \le N_0$.

(2) *There exist processes* $\xi,\zeta : \Omega\times[0,T]\times V^1 \to \mathbf{R}_+$ *that are* $\mathcal{P}_T\otimes\mathcal{V}^1$*measurable and satisfy*

$$|r_1^{-\xi_t(z)} H_t^1(z)|_{\tilde\beta\vee 1} + |r_1^{-\xi_t(z)} \rho_t^1(z)|_{\tilde\beta} + |r_1^{-\xi_t(z)} h_t(z)|_{\tilde\beta} \le \zeta_t(z),$$

for all $(\omega,t,z) \in \Omega\times[0,T]\times V^1$.

We now state our existence and uniqueness theorem for (3.1.1).

Theorem 3.2.5. *Let Assumptions 3.2.3*($\bar\beta$) *and 3.2.4*($\tilde\beta$) *hold for some* $\bar\beta > 1\vee\alpha$ *and* $\tilde\beta > \alpha$. *For any stopping time* $\tau \le T$ *and* $\mathcal{F}_\tau\otimes\mathcal{B}(\mathbf{R}^{d_1})$*-measurable random field* φ *such that for some* $\beta' \in (\alpha,\bar\beta\wedge\tilde\beta)$ *and* $\theta' \ge 0$, $\mathbf{P}$*-a.s.* $r_1^{-\theta'}\varphi \in C^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$, *there exists a unique solution* $u = u(\tau)$ *of (3.1.1) in* $\mathfrak{C}^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$.

3.3 Proof of main theorems

We will first prove uniqueness of the solution of (3.2.1) in the class $\mathfrak{C}^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$. The existence part of the proof of Theorem 3.2.2 is divided into a series of steps. In the first step, by appealing to the representation theorem we derived for solutions of continuous SPDEs shown in Theorem 2.4 in [LM14c], we use an interlacing procedure and the strong limit theorem given in Theorem 2.3 in [LM14c] to show that the space inverse of the flow generated by a jump SDE (i.e. the SDE (3.2.2) without the uncorrelated noise) solves a degenerate linear SIDE. Then we linearly transform the inverse flow of a jump SDE to obtain solutions of degenerate linear SIDEs with free and zero-order terms and an initial condition. In the last step of the proof of Theorem 3.2.2, we introduce an independent Wiener process and Poisson random measure as explained above, apply the results we know for fully degenerate equations, and then take the optional projection. In the last section, Subsection 3.3.4, we prove Theorem 3.2.5 using an interlacing procedure and removing the extra terms in $\hat{b}, \hat{c}$ and $\hat{f}$. The uniqueness of the solution u of (3.1.1) follows directly from our construction.

3.3.1 Proof of uniqueness for Theorem 3.2.2

Proof of Uniqueness for Theorem 3.2.2. Fix a stopping time $\tau \le T$ and $\mathcal{F}_\tau \otimes \mathcal{B}(\mathbf{R}^{d_1})$-measurable random field φ such that for some $\beta' \in (\alpha, \bar{\beta} \wedge \tilde{\beta})$ and $\theta' \ge 0$, $\mathbf{P}$-a.s. $r_1^{-\theta'}\varphi \in C^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$. In this section, we will drop the dependence of processes on t, x, and z when we feel it will not obscure the argument. Let $\hat{u}^1(\tau)$ and $\hat{u}^2(\tau)$ be solutions of (3.2.1) in $\mathfrak{C}^{\beta'}$. It follows that $v := \hat{u}^1(\tau) - \hat{u}^2(\tau)$ solves

$$
\begin{aligned}
dv_t^l &= [(\mathcal{L}_t^{1;l} + \mathcal{L}_t^{2;l})v_t + \hat{b}_t^i \partial_i v_t^l + \hat{c}_t^{l\bar{l}} v_t^{\bar{l}}]dt + \mathcal{N}_t^{1;\varrho} v_t^l dw_t^{1;\varrho} \\
&\quad + \int_{Z^1} \mathcal{I}_{t,z}^{1;l} v_{t-}[\mathbf{1}_{D^1}(z) q^1(dt,dz) + \mathbf{1}_{E^1}(z) p^1(dt,dz)], \quad \tau < t \le T, \\
v_t^l &= 0, \quad t \le \tau, \quad l \in \{1,\dots,d_2\},
\end{aligned}
$$

and $\mathbf{P}$-a.s.

$$
\mathbf{1}_{[\tau_n,\tau_{n+1})}(\cdot) r_1^{-\lambda_n} v \in D([0,T]; C^{\beta'}(\mathbf{R}^{d_1}, \mathbf{R}^{d_2})),
$$

where $(\tau_n)_{n\ge 0}$ is an increasing sequence of $\mathbf{F}$-stopping times with $\tau_0 = 0$ and $\tau_n = T$ for sufficiently large n, and where for each n, λ_n is a positive $\mathcal{F}_{\tau_n}$-measurable random variable. Clearly, it suffices to take $\tau_1 = \tau$ and $\lambda_0 = 0$. Thus, $v_t(x) = 0$ for all $(\omega, t) \in [[\tau_0, \tau_1))$. Assume that for some n, $\mathbf{P}$-a.s. for all t and x, $v_{t\wedge\tau_n}(x) = 0$. We will show that $\mathbf{P}$-a.s. for all

t and x, $\tilde{v}_t(x) := v_{(\tau_n\vee t)\wedge\tau_{n+1}}(x) = 0$. Applying Itô's formula, for all x, **P**-a.s. for all t, we find

$$\begin{aligned} d|\tilde{v}_t|^2 &= \left(2\tilde{v}_t^l\mathfrak{L}_t^{1;l}\tilde{v}_t + |\mathcal{N}_t^1\tilde{v}_t|^2 + 2\tilde{v}_t^l b_t^i\partial_i\tilde{v}_t^l + 2\tilde{v}_t^l c_t^{l\bar{l}}\tilde{v}_t^{\bar{l}}\right)dt \\ &\quad + \left(2\tilde{v}_t^l\mathfrak{I}_{t,z}^{1;l}\tilde{v}_t + \int_{D^1\cup E^1}|\mathcal{I}_{t,z}^{1;l}\tilde{v}_t|^2\pi^1(dz)\right)dt \\ &\quad + \left(2v_t^l\mathfrak{L}_t^{2;l}\tilde{v}_t + 2\tilde{v}_t^l\mathfrak{I}_{t,z}^{2;l}\tilde{v}_t\right)dt + 2v_t^l\mathcal{N}_t^{1;\varrho}\tilde{v}_t^l dw_t^{1;\varrho} \\ &\quad + \int_{Z^1}\left(2\tilde{v}_{t-}^l\mathcal{I}_{t,z}^{1;l}\tilde{v}_{t-} + |\mathcal{I}_{t,z}^{1;l}\tilde{v}_{t-}|^2\right)q^1(dt,dz), \quad \tau_n < t \le \tau_{n+1}, \\ |\tilde{v}_t|^2 &= 0, \quad t\le\tau_n, \quad l\in\{1,\dots,d_2\}, \end{aligned} \tag{3.3.1}$$

where for $\phi\in C_c^\infty(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$, $k\in\{1,2\}$, and $l\in\{1,\dots,d_2\}$,

$$\mathfrak{L}^{k;l}\phi := \frac{1}{2}\sigma^{k;i\varrho}\sigma^{k;j\varrho}\partial_{ij}\phi^l + \sigma^{k;j\varrho}\partial_j\sigma^{k;i\varrho}\partial_i\phi^l + \sigma^{k;i\varrho}\upsilon^{k;l\bar{l}\varrho}\partial_i\phi^{\bar{l}} + \sigma^{k;j\varrho}\partial_j a^{k;l\bar{l}\varrho}\phi^{\bar{l}}$$

and

$$\begin{aligned} \mathfrak{I}^{k;l}\phi :&= \int_{D^k}\left(\rho^{k;l\bar{l}}\phi^{\bar{l}}(\tilde{H}^k) - \rho^{k;l\bar{l}}(\tilde{H}^{k;-1})\phi^{\bar{l}}\right)\pi^k(dz) \\ &\quad + \int_{D^k}\left(\phi^l(\tilde{H}^k) - \phi^l + \mathbf{1}_{(1,2]}(\alpha)F^{k;i}\partial_i\phi^l\right)\pi^k(dz) \\ &\quad + \int_{E^k}\left((I_{d_2}^{l\bar{l}} + \rho^{k;l\bar{l}})\phi^{\bar{l}}(\tilde{H}^k) - \phi^l\right)\pi^k(dz). \end{aligned}$$

For each ω and t, let

$$Q_t = \int_{\mathbf{R}^{d^1}}|\tilde{v}_t(x)|^2 r_1^{-\lambda}(x)dx,$$

where $\lambda = \lambda_n + (d'+2)/2$ and $d' > d_1$. Note that

$$\mathbf{E}Q_t \le \int_{\mathbf{R}^{d^1}} r_1^{-d'}(x)dx\mathbf{E}|r_1^{-\lambda_n}\tilde{v}_t|_0 < \infty.$$

It suffices to show that $\sup_{t\le T}\mathbf{E}Q_t = 0$. To this end, we will multiply the equation (3.3.1) by the weight $r_1^{-2\lambda} = r_1^{-2\lambda_n+1}r_1^{-d'}$, integrate in x, and change the order of the integrals in time and space. Thus, we must verify the assumptions of stochastic Fubini theorem hold (see Corollary 3.4.13 and Remark 3.4.14 as well) with the finite measure $\mu(dx) = r_1^{-d'}(x)dx$ on $\mathbf{R}^{d_1}$. Since b and σ^k have linear growth an υ^k and c are bounded, owing to Lemma 3.4.6, we easily obtain that there is a constant $N = N(d_1,d_2,N_0,\lambda_n)$ such that **P**-a.s for all t,

$$\int_{\mathbf{R}^{d_1}}\left(\sum_{k=1}^{2}2|r_1^{-\lambda_n}\tilde{v}||r_1^{-\lambda_n-2}\mathfrak{L}^k\tilde{v}| + |r_1^{\lambda_n-1}\mathcal{N}^1\tilde{v}|^2\right)r_1^{-d'}dx \le N\sup_{t\le T}|r_1^{-\lambda_n}\tilde{v}|_{\beta'}^2,$$

$$\int_{\mathbf{R}^{d_1}} 4|r_1^{-\lambda_n}\tilde{v}|^2|r_1^{-\lambda_n-1}\mathcal{N}^1\tilde{v}|^2 r_1^{-d'}dx \le N\sup_{t\le T}|r_1^{-\lambda_n}\tilde{v}_t|_{\beta'}^4,$$

and

$$\int_{\mathbf{R}^{d_1}}\left(2|r_1^{-\lambda_n}\tilde{v}|r_1^{-\lambda_n-1}b\partial_i\tilde{v}|+2|r_1^{-\lambda_n}\tilde{v}||r_1^{-\lambda_n}c\tilde{v}|\right)r_1^{-d'}dx \le N\sup_{t\le T}|r_1^{-\lambda_n}\tilde{v}_t|_{\beta'}^2.$$

For all $\phi\in C^{\alpha}_{loc}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$ and all k,ω,t,x,p, and z,

$$\begin{aligned}
&r_1^{-p}(\phi(\tilde{H}^k)-\phi+\mathbf{1}_{(1,2]}(\alpha)F^{k;i}\partial_i\phi)\\
&=\bar{\phi}(\tilde{H}^k)-\bar{\phi}-\mathbf{1}_{(1,2]}(\alpha)H^{k;i}\partial_i\bar{\phi}+\mathbf{1}_{(1,2]}(\alpha)(H^{k;i}+F^{k;i})\partial_i\bar{\phi}\\
&+p\mathbf{1}_{(1,2]}(\alpha)(H^{k;i}+F^{k;i})r_1^{-2}x^i\bar{\phi}+\left(\frac{r_1^p(\tilde{H}^k)}{r_1^p}-1\right)(\bar{\phi}(\tilde{H}^k)-\mathbf{1}_{(1,2]}(\alpha)\bar{\phi})\\
&+\mathbf{1}_{(1,2]}(\alpha)\left(\frac{r_1^p(\tilde{H}^k)}{r_1^p}-1+pH^{k;i}r_1^{-2}x^i\right)\bar{\phi},
\end{aligned}\tag{3.3.2}$$

where $\bar{\phi}:=r^{-p}\phi$. By Taylor's formula, for all $\phi\in C^{\alpha}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$ and all k,ω,t,x, and z, we have

$$|\phi(\tilde{H}^k)-\phi-\mathbf{1}_{(1,2]}(\alpha)H^{k;i}\partial_i\phi|\le r_1^{\alpha}|\phi|_{\alpha}|r_1^{-1}H|_0^{\alpha}.\tag{3.3.3}$$

Combining (3.3.2), (3.3.3), and the estimates given in Lemma 3.4.10 (1), for all k,ω,t,x and z, we obtain

$$r_1^{-\alpha}|\rho^k(\tilde{H}^{k;-1})-\rho^k|\le N|\rho|_{\alpha\wedge 1}|r_1^{-1}H^k|_0^{\alpha\wedge 1}$$

and

$$\begin{aligned}
&r_1^{-\lambda_n-\alpha}|\tilde{v}(\tilde{H}^k)-\tilde{v}+\mathbf{1}_{(1,2]}(\alpha)F^{k;i}\partial_i\tilde{v}|\\
&\le N|r_1^{-\lambda_n}\tilde{v}|_{\alpha}\left(|r_1^{-1}H^k|_0^{\alpha}+|r_1^{-1}H|_0[H^k]_1+|r_1^{-1}H|_0^{[\alpha]^-+1}+[H]_1^{[\alpha]^-+1}\right),
\end{aligned}\tag{3.3.4}$$

for some constant $N=N(d_1,\lambda_n,N_0,\eta^1,\eta^2)$. Therefore, $\mathbf{P}$-a.s for all t,

$$\int_{\mathbf{R}^{d_1}}\left(\sum_{k=1}^{2}2|r_1^{-\lambda_n}\tilde{v}||r_1^{-\lambda_n-2}\mathfrak{I}^k\tilde{v}|+\int_{D^1\cup E^1}|r_1^{-\lambda-1}\mathcal{I}_z\tilde{v}|^2\pi^1(dz)\right)r_1^{-d'}dx\le N\sup_{t\le T}|r_1^{-\lambda_n}\tilde{v}|_{\beta'}^2,$$

and

$$\int_{\mathbf{R}^{d_1}}\left(2|r_1^{-\lambda_n}\tilde{v}||r_1^{-\lambda_n-2}\mathcal{I}_z^k\tilde{v}|+|r_1^{-\lambda_n-1}\mathcal{I}_z\tilde{v}|^2\right)^2 r_1^{-d'}dx\le N\sup_{t\le T}|r_1^{-\lambda_n}\tilde{v}_t|_{\beta'}^4,$$

for some constant $N=N(d_1,d_2,\lambda_n,N_0,\eta^1,\eta^2)$.

Let $L^2(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$ be the space of square-integrable functions $f:\mathbf{R}^{d_1}\to\mathbf{R}^{d_2}$ with norm $\|\cdot\|_0$ and inner product $(\cdot,\cdot)_0$. Moreover, let $L^2(\mathbf{R}^{d_1};\ell_2(\mathbf{R}^{d_2}))$ be the space of square-integrable functions $f:\mathbf{R}^{d_1}\to\ell_2(\mathbf{R}^{d_2})$ with norm $\|\cdot\|_0$. With the help of the above estimates and Corollary 3.4.13, denoting $\bar{v} = r^{-\lambda}\tilde{v}$, **P**-a.s. for all t, we have

$$
\begin{aligned}
d\|\bar{v}_t\|_0^2 &= \Big(2(\bar{v}_t^l, \bar{\mathfrak{L}}_t^1 \bar{v}_t)_0 + \|\bar{\mathcal{N}}_t^1 \bar{v}_t\|_0^2 + 2(\bar{v}_t, \bar{\mathfrak{I}}_{t,z}^1 \bar{v}_t)_0 + \int_{D^1\cup E^1} \|\bar{\mathcal{I}}_{t,z}^1 \bar{v}_t\|_0^2 \pi^1(dz)\Big) dt \\
&\quad + \Big(2(\tilde{v}_t, b_t^i \partial_i \tilde{v}_t + \bar{c}_t^{\bar{l}} \tilde{v}_t^{\bar{l}})_0 + 2(\tilde{v}_t, \bar{\mathfrak{L}}_t^2 \tilde{v}_t)_0 + 2(\tilde{v}_t, \bar{\mathfrak{I}}_{t,z}^2 \tilde{v}_t)_0\Big) dt + 2(v_t, \bar{\mathcal{N}}_t^{1;\varrho} \tilde{v}_t)_0 dw_t^{1;\varrho} \\
&\quad + \int_{Z^1} \Big(2(\tilde{v}_{t-}, \bar{\mathcal{I}}_{t,z}^1 \tilde{v}_{t-})_0 + \|\bar{\mathcal{I}}_{t,z}^1 \tilde{v}_{t-}\|_0^2\Big) q^1(dt,dz), \quad \tau_n < t \le \tau_{n+1}, \\
\|\bar{v}_t\|_0^2 &= 0, \quad t \le \tau_n, \quad l \in \{1,\dots,d_2\},
\end{aligned} \tag{3.3.5}
$$

where all coefficients and operators are defined as in (3.2.1) with the following changes:

(1) for each $k \in \{1,2\}$, υ^k is replaced with

$$
\bar{\upsilon}^{k;l\bar{l}} := \upsilon^{k;l\bar{l}} + \mathbf{1}_{\{2\}}(\alpha)\lambda\sigma^{k;i\varrho} r_1^{-2} x^i \delta_{l\bar{l}};
$$

(2) for each $k \in \{1,2\}$, ρ^k replaced with

$$
\bar{\rho}^{k;l\bar{l}} := \rho^{k;l\bar{l}} + \left(\frac{r_1^\lambda(\tilde{H}^k)}{r_1^\lambda} - 1\right)(I_{d_2}^{l\bar{l}} + \rho^{k;l\bar{l}});
$$

(3) c is replaced with

$$
\begin{aligned}
\bar{c}^{l\bar{l}} &= c^{l\bar{l}} + \lambda b^i r^{-2} x^i \delta_{l\bar{l}} + \sum_{k=1}^{2} \lambda^2 \sigma^{k;i\varrho}\sigma^{k;j\varrho} r_1^{-4} x^i x^j \\
&\quad + \sum_{k=1}^{2} \int_{D^k} \left(\left(\frac{r_1^\lambda}{r_1^\lambda(\tilde{H}^{k;-1})} - 1 \right) (I_m^{l\bar{l}} + \rho^k(\tilde{H}^{k;-1})) - \mathbf{1}_{(1,2]}(\alpha)\lambda r_1^{-2} x_i H^{k;i}(\tilde{H}^{k;-1}) \right) \pi^k(dz).
\end{aligned}
$$

Since for all k,ω and t, $|r_1^{-1}\sigma^k|_0 + |r_1^{-1}\nabla\sigma^k|_{\bar{\beta}-1} + |\upsilon^k|_{\bar{\beta}} \le N_0$, for $\bar{\beta} > 1 \vee \alpha$ and $\tilde{\beta} > \alpha$, it is clear that $|\bar{\upsilon}^k|_\alpha \le N$. Moreover, since for all k,ω and t, $|r_1^{-1}H^k|_0 + |H^k|_{\bar{\beta}} \le K^k$ and $|\rho|_{\bar{\beta}'} \le l^k$, applying the estimates in Lemma (3.4.10) (1), we get

$$
|\bar{\rho}^k|_\alpha \le l^k + K^k(1+l^k) \quad \text{and} \quad |c|_\alpha \le N_0.
$$

We will now estimate the drift terms of (3.3.5) in terms of $\|\bar{v}_t\|_0^2$. We write $f \sim g$ if $\int_{\mathbf{R}^{d_1}} |f(x)|dx = \int_{\mathbf{R}^{d_1}} |g(x)|dx$ and $f \ll g$ if $\int_{\mathbf{R}^{d_1}} |f(x)|dx \le \int_{\mathbf{R}^{d_1}} |g(x)|dx$. Using the divergence theorem, for any $v : \mathbf{R}^{d_1} \to \mathbf{R}^{d_2}$, $\sigma : \mathbf{R}^{d_1} \to \mathbf{R}^{d_1}$ and $\upsilon : \mathbf{R}^{d_1} \to \mathbf{R}^{2d_2}$ and all x, we get

$$
\sigma^i\sigma^j v^l v_{ij}^l \sim \frac{1}{2}(\sigma^i\sigma^j)_{ij} v - \sigma^i\sigma^j v_i^l v_j^l = (\sigma_{ij}^i \sigma^j + \sigma_j^i \sigma_i^j)|v|^2 - \sigma^i\sigma^j v_i^l v_j^l,
$$

$$
2\sigma_j^i \sigma^j v^l v_i^l \sim -(\sigma_j^i\sigma^j)_i |v|^2 = (\sigma_{ij}^i \sigma^j + \sigma_j^i \sigma_i^j)|v|^2,
$$

and

$$\sigma^i v^l \upsilon^{l\bar{l}} v_i^{\bar{l}} + \sigma^i v^{\bar{l}} \upsilon^{l\bar{l}} v_i^l = \sigma^i v^l \upsilon_{\mathrm{sym}}^{l\bar{l}} v_i^{\bar{l}} \sim -(\sigma^i \upsilon_{\mathrm{sym}}^{l\bar{l}})_i |v|^2 = -(\sigma_i^i \upsilon_{\mathrm{sym}}^{l\bar{l}} + \sigma^i \upsilon_{\mathrm{sym}}^{l\bar{l}})|v|^2,$$

where $\upsilon_{\mathrm{sym}}^{l\bar{l}} = (\upsilon^{l\bar{l}} + \upsilon^{\bar{l}l})/2$. Consequently, for all ω, t, and x, we have

$$\begin{aligned} 2\bar{v}^l \bar{\mathfrak{L}}^{1;l} \bar{v} + |\bar{\mathcal{N}}^1 \bar{v}|^2 &\sim \frac{1}{2}\left(|\operatorname{div}\sigma^1|^2 - \partial_i \sigma^{1;j\varrho}\partial_j \sigma^{1;i\varrho}\right)|\bar{v}|^2 \\ &\quad - \bar{\upsilon}_{\mathrm{sym}}^{1;l\bar{l}\varrho} \bar{v}^l \bar{v}^{\bar{l}} \operatorname{div}\sigma^{1;\varrho} + |\bar{\upsilon}^1 \bar{v}|^2 \ll N|\bar{v}|^2 \end{aligned}$$

and

$$2\bar{v}^l \bar{\mathfrak{L}}^{(2);l} \bar{v} \ll -(1+\epsilon)|\sigma^{2;i}\partial_i \bar{v}|^2 + N|\bar{v}|^2,$$

for any $\epsilon > 0$, where in the last estimate we have also used Young's inequality. By Lemma 3.4.10 (2) and basic properties of the determinant, there is a constant $N = N(d, N_0, \eta^1, \eta^2)$ such that for all k, ω, t, x, and z,

$$\det \tilde{H}^{k;-1} - 1 = \det(I_{d_1} + F^k) - 1 \leq |\nabla F^k| \leq N|\nabla H^k|$$

and

$$\det \tilde{H}^{k;-1} - 1 - \operatorname{div} F^k \leq |\nabla F^k|^2 \leq N|\nabla H^k|^2.$$

Thus, integrating by parts, for all ω, t, and x, we get

$$\begin{aligned} &2\bar{v}^l \bar{\mathfrak{I}}^{1;l}\bar{v} + \int_{D^1 \cup E^1} |\bar{I}^1 \bar{v}|^2 \pi^1(dz) \sim 2\int_{D^1} \bar{\rho}_{\mathrm{sym}}^{1;l\bar{l}}(\tilde{H}^{1;-1})(\det \nabla \tilde{H}^{1;-1} - 1)\pi^1(dz)\bar{v}^{\bar{l}}\bar{v}^l \\ &\qquad + \int_{D^1 \cup E^1} \left(\det \nabla \tilde{H}^{1;-1} - 1 + \mathbf{1}_{(1,2]}(\alpha)\mathbf{1}_{D^1} \operatorname{div} F^1\right)\pi^1(dz)|\bar{v}|^2 \\ &\qquad + \int_{D^1 \cup E^1} \left(\mathbf{1}_{E^1} 2\bar{\rho}_{\mathrm{sym}}^{1;l\bar{l}}(\tilde{H}^{1;-1})\bar{v}^{\bar{l}}\bar{v}^l + |\bar{\rho}^1(\tilde{H}^{1;-1})\bar{v}|^2\right)\det \nabla \tilde{H}^{1;-1}\pi^1(dz) \\ &\ll N\left(\int_{D^1}\left(K^1(z)^2 + l^1(z)K^1(z) + l^1(z)^2\right)\pi^1(dz) + \int_{E^1}\left(K^k(z) + l^k(z)\right)\pi^1(dz)\right)|\bar{v}|^2. \end{aligned}$$

Analogously, for all ω, t, and x, we attain

$$2\bar{v}^l \bar{\mathfrak{I}}^{2;l}\bar{v} \leq -(1+\epsilon)\int_{D^2 \cup E^2} |\bar{v}(\tilde{H}^2) - \bar{v}|^2 \pi^2(dz) + N|\bar{v}|^2.$$

Therefore, combining the above estimates, **P**-a.s. for all t,

$$Q_t \leq N\int_0^t Q_s ds + M_t, \tag{3.3.6}$$

where $(M_t)_{t\leq T}$ is a càdlàg square-integrable martingale. Taking the expectation of (3.3.6) and applying Gronwall's lemma, we get $\sup_{t\le T} \mathbf{E}Q_t = 0$, which implies that $\mathbf{P}$-a.s. for all t and x, $\tilde{v}_t(x) = 0$. This completes the proof. □

3.3.2 Small jump case

Set $(w^{\varrho})_{\varrho\in\mathbf{N}} = (w^{1;\varrho})_{\varrho\in\mathbf{N}}$, $(Z, \mathcal{Z}, \pi) = (\mathcal{Z}^1, \mathcal{Z}^1, \pi^1)$, $p(dt, dz) = p^1(dt, dz)$, and $q(dt, dz) = q^1(dt, dz)$. Let $\sigma_t(x) = (\sigma_t^{i\varrho}(x))_{1\le i\le d_1, \varrho\ge 1}$ be a $\ell_2(\mathbf{R}^{d_1})$-valued $\mathcal{R}_T \otimes \mathcal{B}(\mathbf{R}^{d_1})$-measurable function defined on $\Omega \times [0, T] \times \mathbf{R}^{d_1}$ and $H_t(x, z) = (H_t^i(x, z))_{1\le i\le d_1}$ be a $\mathcal{P}_T \otimes \mathcal{B}(\mathbf{R}^{d_1}) \otimes \mathcal{Z}$-measurable function defined on $\Omega \times [0, T] \times \mathbf{R}^{d_1} \times Z$.

We introduce the following assumption for $\beta > 1 \vee \alpha$.

Assumption 3.3.1 (β)**.** *(1) There is a constant $N_0 > 0$ such that for all $(\omega, t) \in \Omega \times [0, T]$,*

$$|r_1^{-1} b_t|_0 + |r_1^{-1} \sigma_t|_0 + |\nabla b_t|_{\beta-1} + |\nabla \sigma_t|_{\beta-1} \le N_0.$$

Moreover, for all $(\omega, t, z) \in \Omega \times [0, T] \times Z$,

$$|r_1^{-1} H_t(z)|_0 \le K_t(z) \quad \text{and} \quad |\nabla H_t(z)|_{\beta-1} \le \bar{K}_t(z),$$

where $K : \Omega \times [0, T] \times Z \to \mathbf{R}_+$ is a $\mathcal{P}_T \otimes \mathcal{Z}$-measurable function satisfying

$$K_t(z) + \bar{K}_t(z) + \int_Z \left(K_t(z)^\alpha + \bar{K}_t(z)^2\right) \pi(dz) \le N_0,$$

for all $(\omega, t, z) \in \Omega \times [0, T] \times Z$.

(2) There is a constant $\eta \in (0, 1)$ such that for all $(\omega, t, x, z) \in \{(\omega, t, x, z) \in \Omega \times [0, T] \times \mathbf{R}^{d_1} \times Z : |\nabla H_t(\omega, x, z)| > \eta\}$,

$$|\,(I_{d_1} + \nabla H_t(x, z))^{-1}\,| \le N_0.$$

Let Assumption 3.3.1(β) hold for some $\beta > 1 \vee \alpha$. Let $\tau \le T$ be a stopping time. Consider the system of SIDEs on $[0, T] \times \mathbf{R}^{d_1}$ given by

$$\begin{aligned} dv_t(x) &= \left(\mathbf{1}_{\{2\}}(\alpha)\frac{1}{2}\sigma_t^{i\varrho}(x)\sigma_t^{j\varrho}(x)\partial_{ij}v_t(x) + \mathfrak{b}_t^i(x)\partial_i v_t(x)\right)dt + \mathbf{1}_{\{2\}}(\alpha)\sigma_t^{i\varrho}(x)\partial_i v_t(x)dw_t^\varrho \\ &\quad + \int_Z (v_{t-}(x + H_t(x, z)) - v_{t-}(x))\,[1_{(1,2]}(\alpha)q(dt, dz) + 1_{[0,1]}(\alpha)p(dt, dz)] \\ &\quad + 1_{(1,2]}(\alpha)\int_Z (v_t(x + H_t(x, z)) - v_t(x) + F_t(x, z)\partial_i v_t(x))\,\pi(dz)dt,\ \ \tau < t \le T, \\ v_t(x) &= x, \quad t \le \tau, \end{aligned} \tag{3.3.7}$$

where

$$\mathfrak{b}_t^i(x) := \mathbf{1}_{[1,2]}(\alpha) b_t^i(x) + \mathbf{1}_{\{2\}}(\alpha)\sigma_t^{j\varrho}(x)\partial_j \sigma_t^{i\varrho}(x)$$

and

$$F_t(x,z) := -H_t(\tilde{H}_t^{-1}(x,z),z).$$

We associate with (3.3.7) the stochastic flow $Y_t = Y_t(\tau,x)$, $(t,x) \in [0,T] \times \mathbf{R}^{d_1}$, generated by the SDE

$$\begin{aligned} dY_t &= -\mathbf{1}_{[1,2]}(\alpha) b_t(Y_t)dt - \mathbf{1}_{\{2\}}(\alpha)\sigma_t^{\varrho}(Y_t)dw_t^{\varrho} \\ &\quad + \int_Z F_t(Y_{t-},z)[1_{(1,2]}(z)q(dt,dz) + 1_{[0,1]}(z)p(dt,dz)], \quad \tau < t \leq T, \\ Y_t &= x, \quad t \leq \tau. \end{aligned} \tag{3.3.8}$$

Owing to parts (1) and (2) of Lemma 3.4.10, for all ω, t, and z, the inverse of the mapping $\tilde{F}_t(x,z) := x + F_t(x,z) = x - H_t(\tilde{H}_t^{-1}(x,z),z)$ is $\tilde{H}_t(x,z) := x + H_t(x,z)$ and there is a constant $N = N(d_1, N_0, \beta, \eta)$ such that for all ω, t, x, y, and z,

$$|r_1^{-1}F_t(z)|_0 \leq NK_t(z), \quad |\nabla F_t(z)|_{\beta-1} \leq K_t(z), \quad |(I_{d_1} + \nabla F_t(x,z))^{-1}| \leq N.$$

Thus, by Theorem 2.1 in [LM14c], there is a modification of the solution of (3.3.8), which we still denote by $Y_t = Y_t(\tau,x)$, that is a $C^{\beta'}_{loc}$-diffeomorphism for any $\beta' \in [1,\beta)$. Moreover, $\mathbf{P}$-a.s. $Y_\cdot(\tau,\cdot), Y_\cdot^{-1}(\tau,\cdot) \in D([0,T]; C^{\beta'}_{loc}(\mathbf{R}^{d_1};\mathbf{R}^{d_1}))$, and $Y_{t-}^{-1}(\tau,\cdot)$ coincides with the inverse of $Y_{t-}(\tau,\cdot)$ for all t. In the proof of the following proposition, we will show that the inverse flow $Y_t^{-1}(\tau)$ solves (3.3.7).

Proposition 3.3.1. *Let Assumption 3.3.1(β) hold for some $\beta > 1 \vee \alpha$. For any stopping time $\tau \leq T$ and $\beta' \in [1 \vee \alpha, \beta)$, $v_t(x) = v_t(\tau,x) = Y_t^{-1}(\tau,x)$ solves (3.3.7) and for all $\epsilon > 0$ and $p \geq 2$, there is a constant $N = N(d_1, p, N_0, T, \beta', \eta, \epsilon)$ such that*

$$\mathbf{E}\left[\sup_{t\leq T} |r_1^{-(1+\epsilon)} v_t(\tau)|_0^p\right] + \mathbf{E}\left[\sup_{t\leq T} |r_1^{-\epsilon}\nabla v_t(\tau)|_{\beta'-1}^p\right] \leq N. \tag{3.3.9}$$

Proof. The estimate (3.3.9) is given in Theorem 2.1 in [LM14c], so we only need to show that $Y_t^{-1}(\tau,x)$ solves (3.3.7). Let $(\delta_n)_{n\geq 1}$ be a sequence such that $\delta_n \in (0,\eta)$ for all n and $\delta_n \to 0$ as $n \to \infty$. It is clear that there is a constant $N = N(N_0)$ such that for all ω and t,

$$\pi(\{z : K_t(z) > \delta_n\}) \leq \frac{N}{\delta_n^{\alpha}}. \tag{3.3.10}$$

For each n, consider the system of SIDEs on $[0,T]\times\mathbf{R}^{d_1}$ given by

$$
\begin{aligned}
dv_t^{(n)}(x) &= \Big(\mathbf{1}_{\{2\}}(\alpha)\frac{1}{2}\sigma_t^{i\varrho}(x)\sigma_t^{j\varrho}(x)\partial_{ij}v_t^{(n)}(x) + \mathfrak{b}_t^i(x)\partial_i v_t^{(n)}(x)\Big)dt \\
&+1_{(1,2]}(\alpha)\int_Z \mathbf{1}_{\{K_t>\delta_n\}}(z)\left(v_t^{(n)}(x+H_t(x,z)) - v_t^{(n)}(x) + F_t^i(x,z)\partial_i v_t^{(n)}(x)\right)\pi(dz)dt \\
&+\int_Z \mathbf{1}_{\{K_t>\delta_n\}}(z)\left(v_{t-}^{(n)}(x+H_t(x,z)) - v_{t-}^{(n)}(x)\right)[1_{(1,2]}(\alpha)q(dt,dz) + 1_{[0,1]}(\alpha)p(dt,dz)], \\
&+\mathbf{1}_{\{2\}}(\alpha)\sigma_t^{i\varrho}(x)\partial_i v_t^{(n)}(x)dw_t^\varrho,\quad \tau<t\le T,\quad v_t^{(n)}(x)=x,\quad t\le\tau,
\end{aligned}
\tag{3.3.11}
$$

and the stochastic flow $Y_t^{(n)} = Y_t^{(n)}(\tau,x)$, $(t,x)\in[0,T]\times\mathbf{R}^{d_1}$, generated by the SDE

$$
\begin{aligned}
dY_t^{(n)} &= -\mathbf{1}_{[1,2]}(\alpha)b_t(Y_t^{(n)})dt - \mathbf{1}_{\{2\}}(\alpha)\sigma_t^\varrho(Y_t^{(n)})dw_t^\varrho \\
&\quad+\int_Z \mathbf{1}_{\{K_t>\delta_n\}}(z)F_t(Y_{t-}^{(n)},z)[1_{(1,2]}(\alpha)q(dt,dz) + 1_{[0,1]}(\alpha)p(dt,dz)],\quad \tau<t\le T, \\
Y_t^{(n)}(x) &= x,\quad t\le\tau.
\end{aligned}
\tag{3.3.12}
$$

Since (3.3.10) holds, we can rewrite equation (3.3.12) as

$$
\begin{aligned}
dY_t^{(n)} &= -\Big(\mathbf{1}_{[1,2]}(\alpha)b_t(Y_t^{(n)}) + 1_{(1,2]}(\alpha)\int_Z \mathbf{1}_{\{K_t>\delta_n\}}(z)F_t(Y_t^{(n)},z)\pi(dz)\Big)dt \\
&\quad- \mathbf{1}_{\{2\}}(\alpha)\sigma_t^\varrho(Y_n(t))dw_t^\varrho + \int_Z \mathbf{1}_{\{K_t>\delta_n\}}(z)F_t(Y_{t-}^{(n)},z)p(dt,dz),\quad \tau<t\le T,
\end{aligned}
\tag{3.3.13}
$$

and (3.3.11) as

$$
\begin{aligned}
dv_t^{(n)}(x) &= \Big(\mathbf{1}_{\{2\}}(\alpha)\frac{1}{2}\sigma_t^{i\varrho}(x)\sigma_t^{j\varrho}(x)\partial_{ij}v_t^{(n)}(x) + \mathfrak{b}_t^i(x)\partial_j\sigma_t^{i\varrho}(x)\Big)dt \\
&\quad+\mathbf{1}_{\{2\}}(\alpha)\sigma_t^{i\varrho}(x)\partial_i v_t^{(n)}(x)dw_t^\varrho + 1_{(1,2]}(\alpha)\int_Z \mathbf{1}_{\{K_t>\delta_n\}}(z)F_t^i(x,z)\pi(dz)\partial_i v_t^{(n)}(x)dt \\
&\quad+\int_Z \mathbf{1}_{\{K_t>\delta_n\}}(z)\left(v_{t-}^{(n)}(x+H_t(x,z)) - v_{t-}^{(n)}(x)\right)p(dt,dz),\quad \tau<t\le T.
\end{aligned}
\tag{3.3.14}
$$

It is well-known that the solution $Y_t^{(n)} = Y_t^{(n)}(x)$ of (3.3.13) can be written as the solution of continuous SDEs with a finite number of jumps interlaced. Indeed, for each n and stopping time $\tau'\le T$, consider the stochastic flow $\tilde{Y}_t^{(n)} = \tilde{Y}_t^{(n)}(\tau',x)$, $(t,x)\in[0,T]\times\mathbf{R}^{d_1}$, generated by the SDE

$$
\begin{aligned}
d\tilde{Y}_t^{(n)} &= -\Big(\mathbf{1}_{[1,2]}(\alpha)b_t(\tilde{Y}_t^{(n)}) + 1_{(1,2]}(\alpha)\int_Z \mathbf{1}_{\{K>\delta_n\}}(t,z)F_t(\tilde{Y}_t^{(n)},z)\pi(dz)\Big)dt \\
&\quad- \mathbf{1}_{\{2\}}(\alpha)\sigma_t^\varrho(\tilde{Y}_t^{(n)})dw_t^\varrho,\ \tau'<t\le T, \\
\tilde{Y}_t^{(n)} &= x,\quad t\le\tau'.
\end{aligned}
$$

By Theorems 2.1 and 2.4 and Remark 2.5, there is a modification of $\tilde{Y}_t^{(n)} = \tilde{Y}_t^{(n)}(\tau', x)$, still denoted $\tilde{Y}_t^{(n)}(\tau', x)$, that is a $C^{\beta'}_{loc}$-diffeomorphism. Furthermore, **P**-a.s. we have that

$$\tilde{Y}_\cdot^{(n)}(\tau', \cdot), \tilde{Y}_\cdot^{(n);-1}(\tau', \cdot) \in C([0,T]; C^{\beta'}_{loc})$$

and $\tilde{v}_t^{(n)} = \tilde{v}_t^{(n)}(\tau', x) = \tilde{Y}_t^{(n);-1}(\tau', x)$ solves the SPDE given by

$$\begin{aligned} d\tilde{v}_t^{(n)}(x) &= \left(\mathbf{1}_{\{2\}}(\alpha)\frac{1}{2}\sigma_t^{i\varrho}(x)\sigma_t^{j\varrho}(x)\partial_{ij}v_t^{(n)}(x) + \mathfrak{b}_t^i(x)\partial_i v_t^{(n)}(x)\right)dt \\ &\quad + \mathbf{1}_{\{2\}}(\alpha)\sigma_t^{i\varrho}(x)\partial_i v_t^{(n)}(x)dw_t^\varrho \\ &\quad + \mathbf{1}_{(1,2]}(\alpha)\int_Z \mathbf{1}_{\{K>\delta_n\}}(t,z)F^i(t,z)\pi(dz)\partial_i v_t^{(n)}(x)dt, \quad \tau' < t \le T, \\ \tilde{v}_t^{(n)}(x) &= x, \quad t \le \tau'. \end{aligned}$$

For each n, let

$$A_t^{(n)} = \int_{]0,t]}\int_Z \mathbf{1}_{\{K_s>\delta_n\}}(z)p(ds,dz), \quad t \ge 0,$$

and define the sequence of stopping times $(\tau_l^{(n)})_{l=1}^\infty$ recursively by $\tau_0^{(n)} = \tau$ and

$$\tau_{l+1}^{(n)} = \inf\left\{t > \tau_l^{(n)} : \Delta A_t^{(n)} \neq 0\right\} \wedge T.$$

Fix some $n \ge 1$. It is clear that **P**-a.s. for all x and $t \in [0, \tau_1^{(n)})$,

$$Y_t^{(n);-1}(\tau, x) = \tilde{Y}_t^{(n);-1}(\tau, x) = \tilde{v}_t^{(n)}(\tau, x)$$

satisfies (3.3.14) up to, but not including time $\tau_1^{(n)}$. Moreover, **P**-a.s. for all x,

$$Y^{(n)}_{\tau_1^{(n)}}(\tau, x) = \tilde{Y}^{(n)}_{\tau_1^n -}(\tau, x) + \int_Z F_{\tau_1^{(n)}}(\tilde{Y}^{(n)}_{\tau_1^{(n)}-}(\tau, x), z)p(\{\tau_1^{(n)}\}, dz),$$

and hence

$$Y^{(n);-1}_{\tau_1^{(n)}}(\tau, x) = \int_Z \tilde{v}^{(n)}_{\tau_1^{(n)}-}(\tau, x + H_{\tau_1^{(n)}}(x,z))p(\{\tau_1^{(n)}\}, dz).$$

Consequently, $v_t^{(n)}(\tau, x) = Y_t^{(n);-1}(\tau, x)$ solves (3.3.14) up to and including time $\tau_1^{(n)}$. Assume that for some $l \ge 1$, $v_t^{(n)}(\tau, x) = Y_t^{(n);-1}(\tau, x)$ solves (3.3.14) up to and including time $\tau_l^{(n)}$. Clearly, **P**-a.s. for all x and $t \in [\tau_l^{(n)}, \tau_{l+1}^{(n)})$, $Y_t^{(n)}(x) = \tilde{Y}_t^{(n)}(\tau_l^{(n)}, Y^{(n)}_{\tau_l^{(n)}-}(x))$, and thus **P**-a.s. for all x and $t \in [\tau_l^{(n)}, \tau_{l+1}^{(n)})$,

$$Y_t^{(n);-1}(x) = \tilde{Y}_t^{(n)}(\tau_l^{(n)}, Y^{(n)}_{\tau_l^{(n)}-}(x)) = \tilde{v}_t^{(n)}(\tau_l^{(n)}, Y^{(n)}_{\tau_l^{(n)}-}(x)).$$

Moreover, $\mathbf{P}$-a.s. for all x,

$$Y_n^{-1}(\tau_{l+1}^n, x) = \int_U \tilde{v}_n(\tau_l^n, \tau_{l+1}^n -, x + H(\tau_{l+1}^n, x, z))p(\{\tau_{l+1}^n\}, dz),$$

which implies that $v_t^{(n)}(\tau, x) = Y_t^{(n);-1}(\tau, x)$ solves (3.3.14) up to and including time τ_{l+1}^n. Therefore, by induction, for each n, $v_t^{(n)}(\tau, x) = Y_t^{(n);-1}(\tau, x)$ solves (3.3.14). It is easy to see that for all ω, t, and z,

$$|r_1^{-1}\mathbf{1}_{\{K_t>\delta_n\}}(z)F_t(z) - r_1^{-1}F_t(z)|_0 + |\mathbf{1}_{\{K_t>\delta_n\}}(z)\nabla F_t(z) - \nabla F_t(z)|_{\beta-1} \leq \mathbf{1}_{\{K_t\leq\delta_n\}}(z)K_t(z)$$

and thus

$$d\mathbf{P}dt - \lim_{n\to\infty}\int_D \mathbf{1}_{\{K\leq\delta_n\}}(t,z)K_t(z)^2\pi(dz) + d\mathbf{P}dt - \lim_{n\to\infty}\int_E \mathbf{1}_{\{K\leq\delta_n\}}(t,z)K_t(z)\pi(dz) = 0.$$

By virtue of Theorem 2.3 in [LM14c], for all $\epsilon > 0$ and $p \geq 2$ we have

$$\lim_{n\to\infty}\left(\mathbf{E}\left[\sup_{t\leq T}|r_1^{-(1+\epsilon)}(Y_t^{(n)}(\tau) - r_1^{-(1+\epsilon)}Y_t(\tau)|_0^p\right] + \mathbf{E}\left[\sup_{t\leq T}|r_1^{-\epsilon}\nabla Y_t^{(n)}(\tau) - r_1^{-\epsilon}\nabla Y_t(\tau)|_{\beta'-1}^p\right]\right) = 0,$$

$$\lim_{n\to\infty}\mathbf{E}\left[\sup_{t\leq T}|r_1^{-(1+\epsilon)}Y_t^{(n);-1}(\tau) - r_1^{-(1+\epsilon)}Y_t^{-1}(\tau)|_0^p\right] = 0$$

and

$$\lim_{n\to\infty}\mathbf{E}\left[\sup_{t\leq T}|r_1^{-\epsilon}\nabla Y_t^{(n);-1}(\tau) - r_1^{-\epsilon}\nabla Y_t^{-1}(\tau)|_{\beta'-1}^p\right] = 0.$$

Then passing to the limit in both sides of (3.3.11) and making use of Assumption 3.3.1(β), the estimate (3.3.4), and basic convergence properties of stochastic integrals, we discover that $v_t(\tau, x) = X_t^{-1}(\tau, x)$ solves (3.3.7) . □

3.3.3 Adding free and zero-order terms

Set $(w^\varrho)_{\varrho\in\mathbf{N}} = (w^{1;\varrho})_{\varrho\in\mathbf{N}}$, $(Z, \mathcal{Z}, \pi) = (\mathcal{Z}^1, \mathcal{Z}^1, \pi^1)$, $p(dt, dz) = p^1(dt, dz)$, and $q(dt, dz) = p^1(dt, dz) - \pi^1(dz)dt$. Also, set $D = D^1$, $E = E^1$, and assume $Z = D \cup E$. Let $\upsilon_t(x) = (\upsilon_t^{l\bar{l}\varrho}(\omega, x))_{1\leq l,\bar{l}\leq d_2,\,\varrho\in\mathbf{N}}$ be a $\ell_2(\mathbf{R}^{2d_2})$-valued $\mathcal{R}_T \otimes \mathcal{B}(\mathbf{R}^{d_1})$-measurable function defined on $\Omega\times[0,T]\times\mathbf{R}^{d_1}$ and $\rho_t(x,z) = (\rho_t^{l\bar{l}}(\omega, x, z))_{1\leq l,\bar{l}\leq d_2}$ be a $\mathcal{P}_T\otimes\mathcal{B}(\mathbf{R}^{d_1})\otimes\mathcal{Z}$-measurable function defined on $\Omega \times [0, T] \times \mathbf{R}^{d_1} \times Z$.

We introduce the following assumptions for $\beta > 1 \vee \alpha$ and $\tilde{\beta} > \alpha$.

Assumption 3.3.2 (β). *(1) There is a constant $N_0 > 0$ such that for all $(\omega, t) \in \Omega\times[0, T]$,*

$$|r_1^{-1}b_t|_0 + |r_1^{-1}\sigma_t|_0 + |\nabla b_t|_{\beta-1} + |\nabla\sigma_t|_{\beta-1} \leq N_0.$$

Moreover, for all $(\omega, t, z) \in \Omega \times [0,T] \times Z$,

$$|r_1^{-1} H_t(z)|_0 \leq K_t(z) \quad \textit{and} \quad |\nabla H_t(z)|_{\beta-1} \leq \bar{K}_t(z),$$

where $K : \Omega \times [0,T] \times Z \to \mathbf{R}_+$ *is a* $\mathcal{P}_T \otimes \mathcal{Z}$*-measurable function satisfying*

$$K_t(z) + \bar{K}_t(z) + \int_D \Big(K_t(z)^\alpha + \bar{K}_t(z)^2\Big)\pi(dz) + \int_E \Big(K_t(z)^{\alpha\wedge 1} + \bar{K}_t(z)\Big)\pi(dz) \leq N_0,$$

for all $(\omega, t, z) \in \Omega \times [0,T] \times Z$.

(2) *There is a constant* $\eta \in (0,1)$ *such that for all* $(\omega, t, x, z) \in \{(\omega, t, x, z) \in \Omega \times [0,T] \times \mathbf{R}^{d_1} \times Z : |\nabla H_t(\omega, x, z)| > \eta\}$,

$$|\left(I_{d_1} + \nabla H_t(x,z)\right)^{-1}| \leq N_0.$$

Assumption 3.3.3 ($\tilde{\beta}$)**.** *There is a constant* $N_0 > 0$ *such that for all* $(\omega, t) \in \Omega \times [0,T]$,

$$|c_t|_{\tilde{\beta}} + |\upsilon_t|_{\tilde{\beta}} + |r_1^{-\theta} f_t|_{\tilde{\beta}} + |r_1^{-\theta} g_t|_{\tilde{\beta}} \leq N_0.$$

Moreover, for all $(\omega, t, z) \in \Omega \times [0,T] \times Z$,

$$|\rho_t(z)|_{\tilde{\beta}} + |r_1^{-\theta} h_t(z)|_{\tilde{\beta}} \leq l_t(z),$$

where $l : \Omega \times [0,T] \times Z \to \mathbf{R}_+$ *is a* $\mathcal{P}_T \otimes \mathcal{Z}$*-measurable function satisfying*

$$l_t(z) + \int_D l_t(z)^2 \pi(dz) + \int_E l_t(z)\pi(dz) \leq N_0.$$

$(\omega, t, z) \in \Omega \times [0,T] \times Z$.

Let Assumptions 3.3.2($\bar{\beta}$) and 3.3.3($\tilde{\beta}$) hold for some $\bar{\beta} > 1 \vee \alpha$ and $\tilde{\beta} > \alpha$. Let $\tau \leq T$ be a stopping time and $\varphi : \Omega \times \mathbf{R}^{d_1} \to \mathbf{R}^{d_2}$ be a $\mathcal{F}_\tau \otimes \mathcal{B}(\mathbf{R}^{d_1})$-measurable random field. Consider the system of SIDEs on $[0,T] \times \mathbf{R}^{d_1}$ given by

$$\begin{aligned} dv_t^l &= \Big(\mathcal{L}_t^l v_t + \hat{\mathfrak{b}}_t^i \partial_i \phi^l + \hat{\mathfrak{c}}_t^{l\bar{l}} \phi^{\bar{l}} + \hat{\mathfrak{f}}_t^l\Big)dt + \Big(\mathcal{N}_t^{l\varrho} v_t + g_t^{l\varrho}\Big)dw_t^\varrho \\ &\quad + \int_Z \Big(\mathcal{I}_{t,z}^l v_{t-} + h_t^l(z)\Big)[\mathbf{1}_D(z)q(dt,dz) + \mathbf{1}_E(z)p(dt,dz)], \quad \tau < t \leq T, \\ v_t^l &= \varphi^l, \quad t \leq \tau, \quad l \in \{1,\dots,d_2\}, \end{aligned} \tag{3.3.15}$$

where for $\phi \in C_c^\infty(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$ and $l \in \{1,\dots,d_2\}$,

$$\mathcal{L}_t^l \phi(x) := \mathbf{1}_{\{2\}}(\alpha)\frac{1}{2}\sigma_t^{i\varrho}(x)\sigma_t^{j\varrho}(x)\partial_{ij}\phi^l(x) + \mathbf{1}_{\{2\}}(\alpha)\sigma_t^{i\varrho}(x)a_t^{l\bar{l}\varrho}(x)\partial_i\phi^{\bar{l}}(x)$$

$$
\begin{aligned}
&+\int_{D^k}\rho_t^{l\bar l}(x,z)\left(\phi^{\bar l}(x+H_t(x,z))-\phi^{\bar l}(x)\right)\pi(dz)\\
&+\int_{D^k}\left(\phi^{l}(x+H_t(x,z))-\phi^{l}(x)-\mathbf{1}_{(1,2]}(\alpha)\partial_i\phi^l(x)H_t^i(x,z)\right)\pi(dz)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{N}_t^{l\varrho}\phi^l(x) &:= \mathbf{1}_{\{2\}}(\alpha)\sigma_t^{i\varrho}(x)\partial_i\phi^l(x)+\upsilon_t^{l\bar l\varrho}(x)\phi^{\bar l}(x),\\
\mathcal{I}_{t,z}^{l}\phi^l(x) &:= (I_{d_2}+\rho_t^{l\bar l}(x,z))\phi^{\bar l}(x+H_t(x,z))-\phi^l(x),
\end{aligned}
$$

and where

$$
\begin{aligned}
\hat{\mathfrak{b}}_t^i(x) &: = \mathbf{1}_{[1,2]}(\alpha)b_t^i(x)+\mathbf{1}_{\{2\}}(\alpha)\sigma_t^{j\varrho}(x)\partial_j\sigma_t^{i\varrho}(x)\\
&\quad+\int_D\left(\mathbf{1}_{(1,2]}(\alpha)H_t^i(x,z)-H_t^i(\tilde H_t^{-1}(x,z),z)\right)\pi(dz),\\
\hat{\mathfrak{c}}_t^{l\bar l}(x) &: = c_t^{l\bar l}(x)+\mathbf{1}_{\{2\}}(\alpha)\sigma_t^{j\varrho}(x)\partial_j\upsilon_t^{l\bar l\varrho}(x)+\int_D\left(\rho_t^{l\bar l}(x,z)-\rho_t^{l\bar l}(\tilde H_t^{-1}(x,z),z)\right)\pi(dz),\\
\hat{\mathfrak{f}}_t^{l}(x) &: = f_t^l(x)+\mathbf{1}_{\{2\}}(\alpha)\sigma_t^{j\varrho}(x)\partial_j g_t^l(x)+\int_D\left(h_t^l(x,z)-h_t^l(\tilde H_t^{-1}(x,z),z)\right)\pi(dz).
\end{aligned}
$$

We associate with (3.3.15) the stochastic flow $X_t=X_t(x)=X_t(\tau,x)$, $(t,x)\in[0,T]\times\mathbf{R}^{d_1}$, given by (3.3.8). Let $\Gamma_t(x)=\Gamma_t(\tau,x)$, $(t,x)\in[0,T]\times\mathbf{R}^{d_1}$, be the solution of the linear SDE given by

$$
\begin{aligned}
d\Gamma_t(x) &= (c_t(X_t(x))\Gamma_t(x)+f_t(X_t(x)))\,dt+(\upsilon_t^{\varrho}(X_t(x))\Gamma_t(x)+g_t^{\varrho}(X_t(x)))\,dw_t^{\varrho}\\
&\quad+\int_Z\rho_t(\tilde H_t^{-1}(X_{t-}(x),z),z)\Gamma_{t-}(x)[\mathbf{1}_D(z)q(dt,dz)+\mathbf{1}_E(z)p(dt,dz)]\\
&\quad+\int_Z h_t(\tilde H_t^{-1}(X_{t-}(x),z),z)[\mathbf{1}_D(z)q(dt,dz)+\mathbf{1}_E(z)p(dt,dz)],\quad \tau<t\le T,\\
\Gamma_t(x) &= 0,\quad t\le\tau.
\end{aligned}
\tag{3.3.16}
$$

Let $\Psi_t(x)=\Psi_t(\tau,x)$, $(t,x)\in[0,T]\times\mathbf{R}^{d_1}$, be the unique solution of the linear SDE given by

$$
\begin{aligned}
d\Psi_t(x) &= c_t(X_t(x))\Psi_t(x)dt+\upsilon_t^{\varrho}(X_t(x))\Psi_t(x)dw_t^{\varrho}\\
&\quad+\int_Z\rho_t(\tilde H_t^{-1}(X_{t-}(x),z),z)\Psi_{t-}(x)[\mathbf{1}_D(z)q(dt,dz)+\mathbf{1}_E(z)p(dt,dz)],\quad \tau<t\le T,\\
\Psi_t(x) &= I_{d_2},\quad t\le\tau.
\end{aligned}
$$

In the following lemma, we obtain p-th moment estimates of the weighted Hölder norms of Γ and Ψ.

Lemma 3.3.2. *Let Assumptions 3.3.2($\bar\beta$) and 3.3.3($\tilde\beta$) hold for some $\bar\beta>1\vee\alpha$ and $\tilde\beta>\alpha$. For any stopping time $\tau\le T$ and $\beta'\in[0,\bar\beta\wedge\tilde\beta)$, there exists a $D([0,T],C^{\beta'}_{loc}(\mathbf{R}^{d_1};\mathbf{R}^{d_2}))$-*

modification of $\Gamma(\tau,)$ *and* $\Psi(\tau)$*, also denoted by* $\bar{\Gamma}(\tau)$ *and* $\Psi(\tau)$*, respectively. Moreover, for all* $\epsilon > 0$ *and* $p \geq 2$*, there is a constant* $N = N(d_1, d_2, p, N_0, T, \beta', \eta, \epsilon, \theta)$ *such that*

$$\mathbf{E}\left[\sup_{t\leq T} |r_1^{-(\theta+\epsilon)}\Gamma_t(\tau)|_{\beta'}^p\right] + \mathbf{E}\left[\sup_{t\leq T} |r_1^{-\epsilon}\Psi_t(\tau)|_{\beta'}^p\right] \leq N. \tag{3.3.17}$$

Proof. Let $\tau \leq T$ be a fixed stopping time and $\beta := \bar{\beta} \wedge \tilde{\beta}$. Estimating (3.3.16) directly and using the Burkholder-Davis-Gundy inequality and Lemma 3.4.1, we get

$$\begin{aligned}\mathbf{E}\left[\sup_{t\leq T} |\Gamma_t(x)|^p\right] &\leq N\mathbf{E}\int \left(|\Gamma_t(x)|^p + r_1(X_t(x))^{p\theta}|r_1^{-\theta} f_t|_0^p\right) dt + \mathbf{E}\left(\int r_1(X_t(x))^{2\theta}|r_1^{-\theta} g_t|_0^2 dt\right)^{p/2}\\
&+\mathbf{E}\left(\int\int_D\int r_1(\tilde{H}_t^{-1}(X_t(x)))^{2\theta}|r_1^{-\theta}h_t|_\infty^2\pi(dz)dt\right)^{p/2}\\
&+\mathbf{E}\int\int_{D\cup E} r_1(\tilde{H}_t^{-1}(X_t(x)))^{p\theta}|r_1^{-\theta}h_t|_\infty^p\pi(dz)dt\\
&+\mathbf{E}\left(\int\int_E r_1(\tilde{H}_t^{-1}(X_t(x)))^{\theta}|r_1^{-\theta}h_t|_\infty\pi(dz)dt\right)^p.\end{aligned}$$

Then using multiplicative decomposition

$$h_t(x, \tilde{H}_t^{-1}(X_{t-}(x), z), z) = r_1(X_{t-}(x))^\theta \frac{r_1(\tilde{H}_t^{-1}(X_{t-}(x), z))^\theta}{r_1(X_t(x))^\theta} \frac{h_t(\tilde{H}_t^{-1}(X_{t-}(x), z), z)}{r_1(\tilde{H}_t^{-1}(X_{t-}(x), z))^\theta},$$

Hölder's inequality, Lemma 3.4.10 (1), Lemma 3.2 in [LM14c], and Gronwall's inequality, we get that for all x and y,

$$\mathbf{E}\left[\sup_{t\leq T} |\Gamma_t(x)|^p\right] \leq N r_1^{-\theta p}(x),$$

where $N = N(d_1, p, N_0, T, \eta, \theta)$ is a positive constant. In a similar way, grouping terms in the obvious manner and using Lemma 3.4.3 and Lemma 3.4.10 (3), we obtain

$$\mathbf{E}\left[\sup_{t\leq T} |\Gamma_t(x) - \Gamma_t(y)|^p\right] \leq N(r_1^{-p\theta}(x) \vee r_1^{-p\theta}(y))|x-y|^{(\beta'\wedge 1)p}.$$

Now, assume that $[\beta]^- \geq 1$. As in the proof of Theorem 3.4 in [Kun04], it follows that $\mathfrak{U}_t = \nabla\Gamma_t(\tau, x)$ solves

$$\begin{aligned}d\mathfrak{U}_t &= (c_t(X_t)\mathfrak{U}_t + \nabla c_t(X_t)\nabla X_t\Gamma_t + \nabla f_t(X_t)\nabla X_t)\, dt\\
&+ \int_Z \rho_t(\tilde{H}_t^{-1}(X_{t-}, z), z)\mathfrak{U}_{t-}[\mathbf{1}_D(z)q(dt, dz) + \mathbf{1}_E(z)p(dt, dz)]\\
&+ \int_Z \nabla\rho_t(\tilde{H}_t^{-1}(X_{t-}, z), z)\nabla[\tilde{H}_t^{-1}(X_{t-})]\Gamma_{t-}[\mathbf{1}_D(z)q(dt, dz) + \mathbf{1}_E(z)p(dt, dz)]\\
&+ \int_Z \nabla h_t(x, \tilde{H}_t^{-1}(X_t, z), z)\nabla[\tilde{H}_t^{-1}(X_{t-})]][\mathbf{1}_D(z)q(dt, dz) + \mathbf{1}_E(z)p(dt, dz)]\end{aligned}$$

$$
\begin{aligned}
&+ \left(\upsilon^{\varrho}_t(X_t)\mathfrak{U}_t + \nabla \upsilon^{\varrho}_t(X_t)\nabla X_t\Gamma_t + \nabla g^{\varrho}_t(X_t)\nabla X_t\right) dw^{\varrho}_t \quad \tau < t \leq T,\\
\mathfrak{U}_t &= 0, \quad t \leq \tau.
\end{aligned}
$$

Recall that by Lemma 3.4.6, a function $\phi : \mathbf{R}^{d_1} \to \mathbf{R}^n$, $n \geq 1$ satisfies $|r^{-\theta}\phi|_\beta < \infty$ if and only if $|r^{-\theta}\phi|_0, \ldots, |r^{-\theta}\partial^\gamma \phi|_0$, $|\gamma| \leq [\beta]^-$, and $[r^{-\theta}\partial^\gamma \phi]|_{\{\beta\}^+}$ are finite. Estimating as above and using Proposition 3.4 in [LM14c] and Lemma 3.4.10 we obtain that for all $p \geq 2$, there is a constant $N = N(d_1, d_2, p, N_0, T, \theta)$ such that for all x and y,

$$
\mathbf{E}\left[\sup_{t\leq T} |\nabla\Gamma_t(x)|^p\right] \leq r_1^{-p\theta}(x)N
$$

and

$$
\mathbf{E}\left[\sup_{t\leq T} |\nabla\Gamma_t(x) - \nabla\Gamma_t(y)|^p\right] \leq N(r_1^{-p\theta}(x) \vee r_1^{-p\theta}(y))|x-y|^{((\beta-1)\wedge 1)p}.
$$

Using induction we get that for all $p \geq 2$ and all multi-indices γ with $0 \leq |\gamma| \leq [\beta]^-$ and all x,

$$
\mathbf{E} \sup_{t\leq T} [|\partial^\gamma \Gamma_t(x)|^p] \leq r_1^{-p\theta}(x)N,
$$

and for all multi-indices γ with $|\gamma| = [\beta]^-$ and all x, y,

$$
\mathbf{E}\left[\sup_{t\leq T} |\partial^\gamma\Gamma_t(x) - \partial^\gamma\Gamma_t(y)|^p\right] \leq N(r_1^{-p\theta}(x) \vee r_1^{-p\theta}(y))|x-y|^{(\beta-[\beta]^-)p},
$$

for a constant $N = N(d_1, d_2, p, N_0, T, \beta, \eta, \theta)$. It is also clear that for all $p \geq 2$ and all multi-indices γ with $0 \leq |\gamma| \leq [\beta]^-$ and all x,

$$
\mathbf{E}\left[\sup_{t\leq T} |\partial^\gamma \Psi_t(x)|^p\right] \leq N,
$$

and for all multi-indices γ with $|\gamma| = [\beta]^-$ and all x, y,

$$
\mathbf{E}\left[\sup_{t\leq T} |\partial^\gamma\Psi_t(x) - \partial^\gamma\Psi_t(y)|^p\right] \leq N|x-y|^{(\beta-[\beta]^-)p}.
$$

We obtain the existence of a $D([0,T], C^{\beta'}_{loc}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))$-modification of $\Gamma(\tau)$ and $\Psi(\tau)$ using estimate (3.3.17) and Corollary 5.3 in [LM14c]. This completes the proof. □

Let $\tilde{\Phi}_t(x) = \tilde{\Phi}_t(\tau, x)$, $(t, x) \in [0, T] \times \mathbf{R}^{d_1}$, be the solution of the linear SDE given by

$$
\begin{aligned}
d\tilde{\Phi}_t(x) &= \left(c_t(X_t(x))\tilde{\Phi}_t(x) + f_t(X_t(x))\right) dt + \left(\upsilon^{\varrho}_t(X_t(x))\tilde{\Phi}_t(x) + g^{\varrho}_t(X_t(x))\right) dw^{\varrho}_t\\
&\quad + \int_Z \rho_t(\tilde{H}^{-1}_t(X_{t-}(x), z), z)\tilde{\Phi}_{t-}(x, y)[\mathbf{1}_D(z)q(dt, dz) + \mathbf{1}_E(z)p(dt, dz)]
\end{aligned}
$$

$$+\int_Z h_t(\tilde{H}_t^{-1}(X_{t-}(x),z),z)[\mathbf{1}_D(z)q(dt,dz)+\mathbf{1}_E(z)p(dt,dz)],\quad \tau<t\le T,$$
$$\tilde{\Phi}_t(x)=\varphi(x),\quad t\le\tau.$$

The following is a simple corollary of Lemma 3.3.2.

Corollary 3.3.3. *Let Assumptions 3.3.2($\bar{\beta}$) and 3.3.3($\tilde{\beta}$) hold for some $\bar{\beta}>1\vee\alpha$ and $\tilde{\beta}>\alpha$. For any stopping time $\tau\le T$ and $\mathcal{F}_\tau\otimes\mathcal{B}(\mathbf{R}^{d_1})$-measurable random field φ such that for some $\beta'\in[0,\bar{\beta}\wedge\tilde{\beta})$, $\mathbf{P}$-a.s. $\varphi\in C^{\beta'}_{loc}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$, there is a $D([0,T];C^{\beta'}_{loc}(\mathbf{R}^{d_1},\mathbf{R}^{d_2}))$-modification of $\tilde{\Phi}(\tau)$, also denoted by $\tilde{\Phi}(\tau)$, and $\mathbf{P}$-a.s. for all $(t,x)\in[0,T]\times\mathbf{R}^{d_1}$,*

$$\tilde{\Phi}_t(\tau,x)=\Psi_t(x)\varphi(x)+\Gamma_t(x).$$

Moreover, if for some $\theta'\ge0$ and $\beta'\in[0,\bar{\beta}\wedge\tilde{\beta})$, $\mathbf{P}$-a.s. $r_1^{-\theta'}\varphi\in C^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$, then for all $\epsilon>0$ and $p\ge2$, there is a constant $N=N(d_1,d_2,p,N_0,T,\theta,\theta',\beta',\epsilon)$ such that

$$\mathbf{E}\left[\sup_{t\le T}|r_1^{-(\theta\vee\theta')-\epsilon}\tilde{\Phi}_t(\tau)|^p_{\beta'}\Big|\mathcal{F}_\tau\right]\le N(|r_1^{-\theta'}\varphi|^p_{\beta'}+1). \tag{3.3.18}$$

Let us now state our main result concerning degenerate SIDEs and their connection with linear transformations of inverse flows of jump SDEs.

Proposition 3.3.4. *Let Assumptions 3.3.2($\bar{\beta}$) and 3.3.3($\tilde{\beta}$) hold for some $\bar{\beta}>1\vee\alpha$ and $\tilde{\beta}>\alpha$. For any stopping time $\tau\le T$ and $\mathcal{F}_\tau\otimes\mathcal{B}(\mathbf{R}^{d_1})$-measurable random field φ such that for some $\beta'\in(\alpha,\bar{\beta}\wedge\tilde{\beta})$ and $\theta'\ge0$, $\mathbf{P}$-a.s. $r_1^{-\theta'}\varphi\in C^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$, we have that $\mathbf{P}$-a.s. $\tilde{\Phi}(\tau,X^{-1}(\tau))\in D([0,T];C^{\beta'}_{loc}(\mathbf{R}^{d_1};\mathbf{R}^{d_2}))$ and $v_t(x)=v_t(\tau,x)=\tilde{\Phi}_t(\tau,X_t^{-1}(\tau,x))$ solves (3.3.15). Moreover, for all $\epsilon>0$ and $p\ge2$,*

$$\mathbf{E}\left[\sup_{t\le T}|r_1^{-(\theta\vee\theta')-\epsilon}v_t(\tau)|^p_{\beta'}\Big|\mathcal{F}_\tau\right]\le N(|r_1^{-\theta'}\varphi|^p_{\beta'}+1), \tag{3.3.19}$$

for a constant $N=N(d_1,d_2,p,N_0,T,\beta',\eta,\epsilon,\theta,\theta')$.

Proof. Fix a stopping time $\tau\le T$ and random field φ such that for some $\beta'\in(\alpha,\bar{\beta}\wedge\tilde{\beta})$ and $\theta'\ge0$, $\mathbf{P}$-a.s. $r_1^{-\theta'}\varphi\in C^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$. By virtue of Corollary 3.3.3 and Theorem 2.1 in [LM14c], $\mathbf{P}$-a.s.

$$\tilde{\Phi}(\tau,X^{-1}(\tau))\in D([0,T];C^{\beta'}_{loc}(\mathbf{R}^{d_1},\mathbf{R}^{d_2})).$$

Then using the Ito-Wenzell formula (Proposition 3.4.16) and following a simple calculation, we obtain that $v_t(\tau,x):=\tilde{\Phi}_t(\tau,X_t^{-1}(\tau,x))$ solves (3.3.15). By Theorem 2.1 in [LM14c] and Corollary 3.3.3, for all $\epsilon>0$ and $p\ge2$, there exists a constant $N=N(d_1,p,N_0,T,$

$\beta', \eta, \epsilon)$ such that

$$\mathbf{E}\left[\sup_{t\le T}|r_1^{-(1+\epsilon)}X_t^{-1}(\tau)|_{\beta'}^p\right]+\mathbf{E}\left[\sup_{t\le T}|r_1^{-\epsilon}\nabla X_t^{-1}(\tau)|_{\beta'-1}^p\right]\le N. \tag{3.3.20}$$

Therefore applying Lemma 3.4.9 and Hölder's inequalty and using the estimates (3.3.20) and (3.3.18), we procure (3.3.19), which completes the proof. □

3.3.4 Adding uncorrelated part (Proof of Theorem 3.2.2)

Proof of Theorem 3.2.2 . Fix a stopping time $\tau \le T$ and random field φ such that for some $\beta' \in (\alpha, \bar\beta \wedge \tilde\beta)$ and $\theta' \ge 0$, **P**-a.s. $r_1^{-\theta'}\varphi \in C^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$. Consider the system of SIDEs given by

$$\begin{aligned}
d\tilde v_t^l &= \left((\mathcal{L}_t^{1;l}+\mathcal{L}_t^{2;l})\tilde v_t + \mathbf{1}_{[1,2]}(\alpha)\hat b_t^i\partial_i u_t^l + \hat c_t^{l\bar l}u_t^{\bar l}(x) + \hat f_t^l\right)dt + \left(\mathcal{N}_t^{1;l\varrho}\tilde v_t + g_t^{l\varrho}\right)dw_t^{1;\varrho}\\
&\quad + \mathcal{N}_t^{2;l\varrho}\tilde v_t dw_t^{2;\varrho} + \int_{Z^1}\left(\mathcal{I}_{t,z}^{1;l}\tilde v_{t-} + h_t^l(z)\right)[\mathbf{1}_{D^1}(z)q^1(dt,dz)+\mathbf{1}_{E^1}p^1(dt,dz)]\\
&\quad + \int_{Z^2}\mathcal{I}_{t,z}^{2;l}\tilde v_{t-}[\mathbf{1}_{D^2}(z)q^2(dt,dz)+\mathbf{1}_{E^2}(z)p^2(dt,dz)] \quad \tau < t \le T,\\
\tilde v_t^l &= \varphi^l, \quad t \le \tau, \quad l \in \{1,\dots,d_2\},
\end{aligned}$$

where for $\phi \in C_c^\infty(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$ and $l \in \{1,\dots,d_2\}$,

$$\begin{aligned}
\mathcal{N}_t^{2;l\varrho}\phi(x) &:= \mathbf{1}_{\{2\}}(\alpha)\sigma_t^{2;i\varrho}(x)\partial_i\phi^l(x) + \upsilon_t^{2;l\bar l\varrho}(x)\phi^{\bar l}(x), \quad \varrho \in \mathbf{N},\\
\mathcal{I}_{t,z}^{2;l}\phi(x) &:= (I_{d_2}^{l\bar l} + \rho_t^{2;l\bar l}(x,z))\phi^{\bar l}(x+H_t^2(x,z)) - \phi^l(x).
\end{aligned}$$

By Proposition 3.3.4, **P**-a.s. $\Phi(\tau, X^{-1}(\tau)) \in D([0,T]; C_{loc}^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2}))$ and $\tilde v_t(\tau,x) = \Phi_t(\tau, X_t^{-1}(\tau,x))$ solves (3.3.15). We write $v_t(x) = v_t(\tau,x)$. Moreover, for all $\epsilon > 0$ and $p \ge 2$,

$$\mathbf{E}\left[\sup_{t\le T}|r_1^{-(\theta\vee\theta')-\epsilon}\tilde v_t(\tau)|_{\beta'}^p\middle|\mathcal{F}_\tau\right] \le N(|r_1^{-\theta'}\varphi|_{\beta'}^p + 1), \tag{3.3.21}$$

where $N = N(d_1, d_2, p, N_0, T, \beta', \eta^1, \eta^2, \epsilon, \theta, \theta')$ is a positive constant. Without loss of generality we will assume that for all ω and t, $|r_1^{-\theta'}\varphi|_{\beta'} \le N$, since we can always multiply the equation by indicator function. For each $n \in \mathbf{N}_0$, let $C_{loc}^n(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$ be the separable Fréchet space of n-times continuously differentiable functions $f : \mathbf{R}^{d_1} \to \mathbf{R}^{d_2}$ endowed with the countable set of semi-norms given by

$$|f|_{n,k} = \sum_{0\le|\gamma|\le n}\sup_{|x|\le k}|\partial^\gamma f(x)|, \quad k \in \mathbf{N}.$$

Owing to Lemma 3.4.2, there is a the family of measures $E^t_\omega(dU)$, $(\omega,t)\in\Omega\times[0,T]$ on $D([0,T];C^{[\beta]^-}_{loc}(\mathbf{R}^{d_1};\mathbf{R}^{d_2}))$, corresponding to $\mathfrak{A}=\tilde{v}$ such that for all bounded $G:\Omega\times[0,T]\times[0,T]\times D([0,T];C^{[\beta]^-}_{loc}(\mathbf{R}^{d_1};\mathbf{R}^{d_2}))\to\mathbf{R}^{d_2}$ that are $\mathcal{O}_{\mathcal{T}}\times\mathcal{B}([0,T])\times\mathcal{B}(D([0,T];C^{[\beta]^-}_{loc}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})))$ measurable, $\mathbf{P}$-a.s. for all t, we have

$$E^t[G_t(t,\tilde{v})]=\int_{D([0,T];C^{[\beta']^-}_{loc}(\mathbf{R}^{d_1};\mathbf{R}^{d_2}))}G_t(t,U)E^t(dU)=\mathbf{E}\left[G_t(t,\tilde{v})|\mathcal{F}_t\right],$$

where the right-hand-side is the càdlàg modification of the conditional expectation. Set

$$\hat{u}_t(x)=\hat{u}_t(\tau,x)=E^t[\tilde{v}_t(\tau,x)]=\int_{D([0,T];C^{[\beta']^-}_{loc}(\mathbf{R}^{d_1};\mathbf{R}^{d_2}))}U_t(x)E^t(dU).$$

Let $\lambda=(\theta\vee\theta')+\epsilon$. We claim that for all multi-indices γ with $|\gamma|\le[\beta]^-$, $\mathbf{P}$-a.s. for all t and x,

$$\partial^\gamma[r_1^{-\lambda}(x)\hat{u}_t(x)]=\int_{D([0,T];C^{[\beta']^-}_{loc}(\mathbf{R}^{d_1};\mathbf{R}^{d_2}))}\partial^\gamma[r_1^{-\lambda}(x)U_t(x)]E^t(dU)=E^t[\partial^\gamma[r_1^{-\lambda}(x)\tilde{v}_t(x)]].$$

Indeed, since

$$M_t=E^t\left[\sup_{s\le T}|\partial^\gamma[r_1^{-\lambda}\tilde{v}_s]|_0\right],\quad t\in[0,T],$$

is a $(\mathbf{F},\mathbf{P})$ martingale, we have

$$\mathbf{E}\left[\sup_{t\le T}|M_t|^2\right]\le 4\mathbf{E}\left[|M_T|^2\right]\le 4\mathbf{E}\left[\sup_{t\le T}|\partial^\gamma[r_1^{-\lambda}\tilde{v}_t]|_0^2|\right]<\infty, \tag{3.3.22}$$

which implies that $\mathbf{P}$-a.s. for all t,

$$\int_{D([0,T];C^{[\beta']^-}_{loc}(\mathbf{R}^{d_1};\mathbf{R}^{d_2}))}\sup_{s\le T,x\in\mathbf{R}^{d_1}}|\partial^\gamma[r_1^{-\lambda}(x)U_s(x)]|E^t(dU)=E^t\left[\sup_{t\le T}|\partial^\gamma[r_1^{-\lambda}\tilde{v}_t]|_0\right]<\infty.$$

Similarly, since $\mathbf{E}\left[\sup_{t\le T}|r_1^{-\lambda}\tilde{v}_t|^2_{\beta'}\right]<\infty$, $\mathbf{P}$-a.s. for all x and y,

$$\begin{aligned}\frac{|\partial^\gamma[r_1^{-\lambda}(x)\hat{u}_t(x)]-\partial^\gamma[r_1^{-\lambda}(y)\hat{u}_t(y)]|}{|x-y|^{\{\beta'\}^+}}&\le E^t\left[\frac{|\partial^\gamma[r_1^{-\lambda}(x)\tilde{v}_t(x)]-\partial^\gamma[r_1^{-\lambda}(y)\tilde{v}_t(y)]|}{|x-y|^{\{\beta'\}^+}}\right]\\&\le E^t[|r_1^{-\lambda}\tilde{v}_t|_{\beta'}],\end{aligned}$$

and thus, $\mathbf{P}$-a.s.

$$\sup_{t\le T}|r_1^{-\lambda}\hat{u}_t|_{\beta'}\le\sup_{t\le T}E^t\left[\sup_{t\le T}|r_1^{-\lambda}\tilde{v}_t|_{\beta'}\right]<\infty.$$

Thus, **P**-a.s. $r_1^{-\lambda}(\cdot)\hat{u}(\tau) \in D([0,T]; C^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))$ and (3.2.4) follows from (3.3.21) (see the argument (3.3.22)). For each $l \in \{1, \ldots, d_2\}$, let $\mathcal{A}_t^l(x) = \mathcal{A}_t^l(z)$ be defined by

$$\begin{aligned}\mathcal{A}_t^l = \varphi^l &+ \int_{]\tau,\tau\vee t]} \left((\mathcal{L}_s^{1;l} + \mathcal{L}_s^{2;l})\hat{u}_s + \mathbf{1}_{[1,2]}(\alpha)\hat{b}_s^i \partial_i \hat{u}_s^l + \hat{c}_s^{l\bar{l}}\hat{u}_s^{\bar{l}} + \hat{f}_s^l \right) ds \\ &+ \int_{]\tau,\tau\vee t]} \left(\mathcal{N}_s^{1;l\varrho}\hat{u}_s + g_s^{l\varrho} \right) dw_s^{1;\varrho} \\ &+ \int_{]\tau,\tau\vee t]} \int_{Z^1} \left(\mathcal{I}_{s,z}^{1;l}\hat{u}_{s-} + h_s^l(z) \right) [\mathbf{1}_{D^1}(z) q^1(ds,dz) + \mathbf{1}_{E^1}(z) p^1(ds,dz)].\end{aligned}$$

By Theorem 12.21 in [Jac79], the representation property holds for $(\mathbf{F}, \mathbf{P})$, and hence every bounded $(\mathbf{F}, \mathbf{P})$- martingale issuing from zero can be represented as

$$M_t = \int_{]0,t]} o_s^{\varrho} dw_s^{1;\varrho} + \int_{]0,t]} \int_{Z^1} e_s(z) q^1(ds,dz), \quad t \in [0,T],$$

where

$$\mathbf{E} \int_{]0,T]} |o_s|^2 ds + \mathbf{E} \int_{]0,T]} \int_{Z^1} |e_s(z)|^2 \pi^1(dz) ds < \infty.$$

Then for an arbitrary $\mathbf{F}$-stopping time $\bar{\tau} \le T$ and bounded $(\mathbf{F}, \mathbf{P})$- martingale, applying Itô's product rule and taking the expectation, we obtain

$$\mathbf{E}\tilde{v}_{\bar{\tau}}(\tau, x)\bar{M}_{\bar{\tau}} = \mathbf{E}\mathcal{A}_{\bar{\tau}}(x)\bar{M}_{\bar{\tau}}.$$

Since the optional projection is unique (see Theorem 13 in Chapter 1, Section 8 in [LS89]), **P**-a.s. for all t and x, $\hat{u}_t(x) = \mathcal{A}_t(x)$. This completes the proof. □

3.3.5 Proof of Theorem 3.2.5

Proof of Theorem 3.2.5. Fix a stopping time $\tau \le T$ and random field φ such that for some $\beta' \in (\alpha, \bar{\beta} \wedge \tilde{\beta})$ and $\theta' \ge 0$, **P**-a.s. $r_1^{-\theta'}\varphi \in C^{\beta'}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$. For any $\delta > 0$, we can rewrite (3.1.1) as

$$\begin{aligned} du_t^l &= \left((\bar{\mathcal{L}}_t^{1;l} + \mathcal{L}_t^{2;l}) u_t + \mathbf{1}_{[1,2]}(\alpha)\bar{b}_t^i \partial_i u_t^l + \bar{c}_t^{l\bar{l}} u_t^{\bar{l}} + f_t^l \right) dt + \left(\mathcal{N}_t^{1;l\varrho} u_t + g_t^{l\varrho} \right) dw_t^{1;\varrho} \\ &\quad + \int_{Z^1} \left(\bar{\mathcal{I}}_{t,z}^{1;l} u_{t-} + \bar{h}_t^l(z) \right) [1_{D^1}(z) q^1(dt,dz) + 1_{E^1}(z) p^1(dt,dz)] \\ &\quad + \int_{Z^1} \left(\mathbf{1}_{(D^1 \cup E^1) \cap \{K_t^1 > \delta\}}(z) + \mathbf{1}_{V^1}(z) \right) \left(\mathcal{I}_{t,z}^{1;l} u_{t-} + h_t^l(z) \right) p^1(dt,dz), \quad \tau < t \le T, \\ u_t^l &= \varphi^l, \quad t \le \tau, \quad l \in \{1, \ldots, d_2\}, \end{aligned} \tag{3.3.23}$$

where for $\phi \in C_c^\infty(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$ and $l \in \{1,\dots,d_2\}$,

$$
\begin{aligned}
\bar{\mathcal{L}}_t^{1;l}\phi(x) := {}& \mathbf{1}_{\{2\}}(\alpha)\frac{1}{2}\sigma_t^{1;i\varrho}(x)\sigma_t^{1;j\varrho}(x)\partial_{ij}\phi^l(x) + \mathbf{1}_{\{2\}}(\alpha)\sigma_t^{k;i\varrho}(x)\upsilon_t^{1;l\bar{l}\varrho}(x)\partial_i\phi^{\bar{l}}(x)\\
&+ \int_{D^1}\bar{\rho}_t^{1;l\bar{l}}(x,z)\left(\phi^{\bar{l}}(x+\bar{H}_t^1(x,z)) - \phi^{\bar{l}}(x)\right)\pi^1(dz)\\
&+ \int_{D^1}\left(\phi^l(x+\bar{H}_t^1(x,z)) - \phi^l(x) - \mathbf{1}_{(1,2]}(\alpha)\bar{H}_t^{1;i}(x,z)\partial_i\phi^l(x)\right)\pi^1(dz),
\end{aligned}
$$

$$
\begin{aligned}
\bar{\mathcal{I}}_{t,z}^1\phi^l(x) &= (I_{d_2}^{l\bar{l}} + \mathbf{1}_{\{K_t^1\le\delta\}}(z)\rho_t^{1;l\bar{l}}(x,z))\phi^{\bar{l}}(x + \mathbf{1}_{\{K_t^1\le\delta\}}(z)H_t^1(x,z)) - \phi^l(x),\\
\bar{H}^1 &:= \mathbf{1}_{\{K_t^1\le\delta\}}H^1,\ \bar{\rho}^1 := \mathbf{1}_{\{K_t^1\le\delta\}}\rho^1,\ \bar{h} := \mathbf{1}_{\{K_t^1\le\delta\}}h,\\
\bar{b}_t^i(x) &:= b_t^i(x) - \int_{D^1\cap\{K_t^1>\delta\}}\mathbf{1}_{(1,2]}(\alpha)H_t^{1;i}(x,z)\pi^1(dz),\\
\bar{c}_t^{l\bar{l}}(x) &:= c_t^{l\bar{l}}(x) - \int_{D^1\cap\{K_t^1>\delta\}}\rho_t^{1;l\bar{l}}(x,z)\pi^1(dz).
\end{aligned}
$$

For an arbitrary stopping time $\tau' \le T$ and $\mathcal{F}_{\tau'}\otimes\mathcal{B}(\mathbf{R}^{d_1})$-measurable random field $\varphi^{\tau'}$: $\Omega\times\mathbf{R}^{d_1}\to\mathbf{R}^{d_2}$ satisfying for some $\theta(\tau')>0$, $\mathbf{P}$-a.s. $r_1^{-\theta(\tau')}\varphi^{\tau'}\in C^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$, consider the system of SIDEs on $[0,T]\times\mathbf{R}^{d_1}$ given by

$$
\begin{aligned}
dv_t^l &= \left((\bar{\mathcal{L}}_t^{1;l} + \mathcal{L}_t^{2;l})v_t + \mathbf{1}_{[1,2]}(\alpha)\bar{b}_t^i\partial_i v_t^l + \bar{c}_t^{l\bar{l}}v_t^{\bar{l}} + f_t^l\right)dt + \left(\mathcal{N}_t^{1;l\varrho}v_t + g_t^{l\varrho}\right)dw_t^{1;\varrho}\\
&\quad + \int_{Z^1}\left(\bar{\mathcal{I}}_{t,z}^{1;l}u_{t-} + \bar{h}_t^l(z)\right)[1_{D^1}(z)q^1(dt,dz) + 1_{E^1}(z)p^1(dt,dz)],\quad \tau'<t\le T,\\
v_t^l &= \varphi^{\tau';l},\quad t\le\tau',\quad l\in\{1,\dots,d_2\}.
\end{aligned}
\tag{3.3.24}
$$

Set $\bar{H}^2 = H^2$ and $\bar{\rho}^2=\rho^2$. In order to invoke Theorem 3.2.2 and obtain a unique solution $v_t = v_t(\tau',x) = v_t(\tau',\varphi^{\tau'},x)$ of (3.3.24), we will show that for all ω and t,

$$
|r_1^{-1}\tilde{b}_t|_0 + |\nabla\tilde{b}_t|_{\bar{\beta}-1} + |\tilde{c}_t|_{\bar{\beta}} + |r^{-\theta}\tilde{f}|_{\bar{\beta}} \le N_0, \tag{3.3.25}
$$

where

$$
\begin{aligned}
\tilde{b}_t^i(x) :&= \mathbf{1}_{[1,2]}(\alpha)\bar{b}_t^i(x) - \sum_{k=1}^2\mathbf{1}_{\{2\}}(\alpha)\sigma_t^{k;j\varrho}(x)\partial_j\sigma_t^{k;i\varrho}(x)\\
&\quad - \sum_{k=1}^2\int_{D^k}\left(\mathbf{1}_{(1,2]}(\alpha)\bar{H}_t^{k;i}(x,z) - \bar{H}_t^{k;i}(\tilde{\bar{H}}_t^{k;-1}(x,z),z)\right)\pi^k(dz),\\
\tilde{c}_t^{l\bar{l}}(x) :&= \bar{c}_t^{l\bar{l}}(x) - \sum_{k=1}^2\mathbf{1}_{\{2\}}(\alpha)\sigma_t^{k;i\varrho}(x)\partial_i\upsilon_t^{k;l\bar{l}\varrho}(x)
\end{aligned}
$$

$$-\sum_{k=1}^{2}\int_{D^k}\Big(\bar\rho_t^{k;l\bar l}(x,z)-\bar\rho_t^{k;l\bar l}(\tilde{\bar H}_t^{k;-1}(x,z),z)\Big)\pi^k(dz),$$
$$\tilde f_t^l(x):=f_t^l(x)-\sigma_t^{1;j\varrho}(x)\partial_j g_t^l(x)-\int_{D^1}\Big(\bar h_t^l(x,z)-\bar h_t^l(\tilde{\bar H}_t^{1;-1}(x,z),z)\Big)\pi^1(dz).$$

Owing to Assumption 3.2.3($\bar\beta,\delta^1$), we easily deduce that there is a constant $N=N(d_1,N_0,\bar\beta)$ such that for each $k\in\{1,2\}$ and all ω and t,

$$|\sigma_t^{k;j\varrho}\partial_j\sigma_t^{k;\varrho}|_{\bar\beta}+|\sigma_t^{k;j\varrho}\partial_j a_t^{k;\varrho}(x)|_{\bar\beta}+|\sigma_t^{1;j\varrho}\partial_j g_t^{\varrho}|_{\bar\beta}\le N,\ \text{if }\alpha=2.$$

Since $|\nabla\bar H_t^1|_0\le\delta$, for any fixed $\eta^1<1$, for all $(\omega,t,x,z)\in\{(\omega,t,x,z)\in\Omega\times[0,T]\times\mathbf{R}^{d_1}\times(D^1\cup E^1):|\nabla\bar H_t^1(\omega,x,z)|>\eta^1\}$,

$$\left|\Big(I_{d_1}+\nabla H_t^1(\omega,x,z)\Big)^{-1}\right|\le\frac{1}{1-\delta}.$$

Appealing to Assumption 3.2.3($\bar\beta,\delta^1$) and applying Lemma 3.4.10, we obtain that there is a constant $N=N(d_1,d_2,N_0)$ such that for each $k\in\{1,2\}$ and all ω,t, and z,

$$\begin{aligned}|\bar H_t^{k;i}(z)-\bar H_t^{k;i}(\tilde{\bar H}_t^{k;-1}(z),z)|_{\bar\beta}&\le N(K_t^k(z)+\bar K_t^k(z))^2+N\mathbf{1}_{(0,1]}(\{\bar\beta\}^++\delta^k)\tilde K_t^k(z)K_t^k(z)^{\delta^k}\\&\quad+N\mathbf{1}_{(1,2]}(\{\bar\beta\}^++\delta^k)\Big(\tilde K_t^k(z)K_t^k(z)^{\delta^k}+\bar K_t^k(z)^2\Big),\\|\bar\rho_t^k(z)-\bar\rho_t^k(\tilde{\bar H}_t^{k;-1}(z),z)|_{\bar\beta}&\le Nl_t^k(z)(K_t^k(z)+\bar K_t^k(z))+N\mathbf{1}_{(0,1]}(\{\bar\beta\}^++\mu^k)\tilde l_t^k(z)K_t^k(z)^{\mu^k}\\&\quad+N\mathbf{1}_{(1,2]}(\{\bar\beta\}^++\mu^k)\Big(\tilde l_t^k(z)K_t^k(z)^{\mu^k}+l_t^k(z)\bar K_t^k(z)\Big),\end{aligned}$$

and

$$\begin{aligned}|r_1^{-\theta}\bar h_t(z)-r_1^{-\theta}\bar h_t(\tilde{\bar H}_t^{1;-1}(z),z)|_{\bar\beta}&\le Nl_t^1(z)(K_t^1(z)+\bar K_t^k(z))+N\mathbf{1}_{(0,1]}(\{\bar\beta\}^++\mu^1)\tilde l_t^k(z)K_t^1(z)^{\mu^1}\\&\quad+N\mathbf{1}_{(1,2]}(\{\bar\beta\}^++\mu^1)\Big(\tilde l_t^k(z)K_t^1(z)^{\mu^1}+l_t^k(z)\bar K_t^1(z)\Big).\end{aligned}$$

Moreover, using Lemma 3.4.10, we find that there is a constant $N=N(d_1,d_2,N_0)$ such that for each $k\in\{1,2\}$ and all ω,t, and z,

$$|r_1^{-1}\bar H_t^k(\tilde{\bar H}_t^{k;-1}(z),z)|_0\le|r_1^{-1}H^k|_0,\quad|\nabla[\bar H_t^{k;i}(\tilde{\bar H}_t^{k;-1}(z),z)]|_{\bar\beta}\le|\nabla H^k|_{\bar\beta-1}.$$

Combining the above estimates and using Hölder's inequality and the integrability properties of $l_t^k(z)$ and $K_t^k(z)$, we obtain (3.3.25). Therefore, by Theorem 3.2.2, for each stopping time $\tau'\le T$ and and $\mathcal{F}_{\tau'}\otimes\mathcal{B}(\mathbf{R}^{d_1})$-measurable random field $\varphi^{\tau'}$ satisfying for some $\theta(\tau')>0$, $\mathbf{P}$-a.s. $r_1^{-\theta(\tau')}\varphi^{\tau'}\in C^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$, there exists a unique solution $v_t(x)=v_t(\tau',\varphi^{\tau'},x)$

of (3.3.24) such that

$$\mathbf{E}\left[\sup_{t\le T}|r_1^{-\theta(\tau')\vee\theta-\epsilon}v_t(\tau')|_{\beta'}^p\Big|\mathcal{F}_{\tau'}\right]\le N(|r_1^{-\theta(\tau')}\varphi^{\tau'}|_{\beta'}^p+1), \tag{3.3.26}$$

where $N=N(d_1,d_2,p,N_0,T,\beta',\eta^1,\eta^2,\epsilon,\theta,\theta(\tau'))$ is a positive constant. Let

$$A_t=\int_{]0,t]}\int_{Z^1}\left(\mathbf{1}_{(D^1\cup E^1)\cap\{K_s^1>\eta^1\}}(z)+\mathbf{1}_{V^1}(z)\right)p^1(ds,dz),\quad t\le T.$$

Define a sequence of stopping times $(\tau_n)_{n\ge 0}$ recursively by $\tau_1=\tau$ and

$$\tau_{n+1}=\inf(t>\tau_n:\Delta A_t\ne 0)\wedge T.$$

We construct the unique solution $u=u(\tau)$ of (3.3.23) in $\mathfrak{C}^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$ by interlacing solutions of (3.3.24) along the sequence of stopping times (τ_n). For $(\omega,t)\in[[0,\tau_1))$, we set $u_t(\tau,x)=v_t(\tau,\varphi,x)$ and note that

$$\mathbf{E}\left[\sup_{t\le\tau_1}|r_1^{-\theta'\vee\theta-\epsilon}u_t(\tau)|_{\beta'}^p\Big|\mathcal{F}_{\tau}\right]\le N(|r_1^{-\theta'}\varphi|_{\beta'}^p+1).$$

For each ω and x, we set

$$\begin{aligned}u_{\tau_1}(x)&=u_{\tau_1-}(x)\\&\quad+\int_{Z^1}\left(\mathbf{1}_{(D^1\cup E^1)\cap\{K^1>\eta^1\}}(\tau_1,z)+\mathbf{1}_{V^1}(z)\right)\left(\mathcal{I}^1_{\tau_1,z}u_{\tau_1-}(x)+h^l_{\tau_1}(x,z)\right)p^1(\{\tau_1\},dz).\end{aligned}$$

By virtue of Lemma 3.4.9, there is a constant $N=N(d_1,d_2,\theta,\theta',\zeta_{\tau_1}(z),\beta')$

$$|u_{\tau_1-}\circ\tilde{H}^1_{\tau_1}(z)\cdot r_1^{-\xi_{\tau_1}(z)(\theta\vee\theta'+\epsilon+\beta')}|_{\beta'}\le N|r_1^{-\theta\vee\theta'-\epsilon}u^l_{\tau_1-}|_{\beta'},$$

and hence

$$|r_1^{-\lambda_1}u_{\tau_1}(x)|_{\beta'}\le N|r_1^{-\theta\vee\theta'-\epsilon}u^l_{\tau_1-}|_{\beta'}+\zeta_{\tau_1}(z),$$

where

$$\lambda_1=(\xi_{\tau_1}(z)(\theta\vee\theta'+1+\epsilon+\beta'))\vee\theta\vee(\theta\vee\theta'+\epsilon).$$

We then proceed inductively, each time making use of the estimate (3.3.26), to obtain a unique solution $u=u(\tau)$ of (3.3.23), and hence (3.1.1), in $\mathfrak{C}^{\beta'}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$. This completes the proof of Theorem 3.2.5. □

3.4 Appendix

3.4.1 Martingale and point measure moment estimates

Set $(Z, \mathcal{Z}, \pi) = (Z^1, \mathcal{Z}^1, \pi^1)$, $p(dt, dz) = p^1(dt, dz)$, and $q(dt, dz) = q^1(dt, dz)$. The following moment estimates are used to derive the estimates of Γ_t and Ψ_t in Lemma 3.3.2. The notation $a \underset{p,T}{\sim} b$ is used to indicate that the quantity a is bounded above and below by a constant depending only on p and T times b.

Lemma 3.4.1. *Let $h : \Omega \times [0, T] \times Z \to \mathbf{R}^{d_1}$ be $\mathcal{P}_T \otimes \mathcal{Z}$-measurable*

(1) For any stopping time $\tau \leq T$ and $p \geq 2$,

$$\mathbf{E}\left[\sup_{t\leq\tau}\left|\int_{]0,\tau]}\int_Z h_s(z)q(ds, dz)\right|^p\right] \underset{p,T}{\sim} \mathbf{E}\left[\int_{]0,\tau]}\int_Z |h_s(z)|^p \,\pi(dz)ds\right] + \mathbf{E}\left[\left(\int_{]0,\tau]}\int_Z |h_s(z)|^2 \,\pi(dz)ds\right)^{p/2}\right].x$$

(2) For any stopping time $\tau \leq T$ and $\bar{p} \geq 1$,

$$\mathbf{E}\left[\sup_{t\leq\tau}\left(\int_{]0,\tau]}\int_Z |h_s(z)|p(ds, dz)\right)^{\bar{p}}\right] \underset{p,T}{\sim} \mathbf{E}\left[\int_{]0,\tau]}\int_Z |h_s(z)|^{\bar{p}} \,\pi(dz)ds\right] + \mathbf{E}\left[\left(\int_{]0,\tau]}\int_Z |h_s(z)|\pi(dz)ds\right)^{\bar{p}}\right],$$

Proof. We will only prove part (2), since part (1) is well-known (see, e.g., [Nov75] or [Kun04]). Assume that $h_t(\omega, z) > 0$ for all ω, t and z. Let

$$A_t = \int_{]0,t]}\int_Z h_s(z)p(ds, dz) \quad \text{and} \quad L_t = \int_{]0,t]}\int_Z h_s(z)\pi(dz)ds, \quad t \leq T.$$

It suffices to prove (2) for $p > 1$, since the case $p = 1$ is obvious. Fix an arbitrary stopping time $\tau \leq T$ and $p > 1$. For all t, we have

$$A_t^p = \sum_{s\leq t}\left[(A_{s-} + \Delta A_s)^p - A_{s-}^p\right].$$

Thus, by the inequality

$$b^p \leq (a + b)^p - a^p \leq p(a + b)^{p-1}b \leq p2^{p-1}[a^{p-1}b + b^p], \quad a, b \geq 0,$$

we get

$$A_t^p \geq \int_{]0,t]}\int_Z h_s(z)^p p(ds, dz)$$

and

$$A_t^p \leq p2^{p-2}\left[\int_{]0,t]}\int_h h_s(z)^p p(ds, dz) + \int_0^t\int_Z A_{s-}^{p-1}h_s(z)p(ds, dz)\right].$$

Since A_t is increasing, we obtain

$$\mathbf{E}\int_{]0,\tau]}\int_Z h_s(z)^p p(ds,dz) \leq \mathbf{E}A_\tau^p \leq p2^{p-2}\mathbf{E}\left[\int_{]0,\tau]}\int_Z h_s(z)^p p(ds,dz) + A_\tau^{p-1}L_\tau\right].$$

It is easy to see that

$$\mathbf{E}[L_\tau^p] = p\mathbf{E}\int_{]0,\tau]} L_s^{p-1}dL_s = p\mathbf{E}\int_{]0,\tau]} L_s^{p-1}dA_s \leq p\mathbf{E}[L_\tau^{p-1}A_\tau].$$

Applying Young's inequality, for any $\varepsilon > 0$, we get

$$A_\tau^{p-1}L_\tau \leq \varepsilon A_\tau^p + \frac{(p-1)^{p-1}}{\varepsilon^{p-1}p^p}L_\tau^p \quad \text{and} \quad L_\tau^{p-1}A_\tau \leq \varepsilon L_\tau^p + \frac{(p-1)^{p-1}}{\varepsilon^{p-1}p^p}A_\tau^p.$$

Combining the estimates for any $\varepsilon_1 \in (0, \frac{1}{p})$, we have

$$\left(\frac{\varepsilon_1^{p-1}p^p(1-p\varepsilon_1)}{p(p-1)^{p-1}}\mathbf{E}L_\tau^p\right) \vee \mathbf{E}\int_{]0,\tau]}\int_Z h_s(z)^p p(ds,dz) \leq \mathbf{E}[A_\tau^p].$$

and for any $\varepsilon_2 \in (0, \frac{1}{p2^{p-2}})$

$$\mathbf{E}[A_\tau^p] \leq \frac{p2^{p-2}}{(1-p2^{p-2}\varepsilon_2)}\mathbf{E}\left[\int_{]0,\tau]}\int_Z h_s(z)^p p(ds,dz) + \frac{(p-1)^{p-1}}{\varepsilon_2^{p-1}p^p}L_\tau^p\right],$$

which completes the proof. □

3.4.2 Optional projection

The following lemma concerning the optional projection plays an integral role in Section 3.3.4 and the proof of Theorem 3.2.2. For more information on the Skorokhod $\mathcal{J}_1$-topology, we refer the reader to Chapter 6, Section 1 of [LS89]. Also, we refer the reader to Theorem 5.3 [Kal97] for the construction of regular conditional probability measures on Borel spaces.

Lemma 3.4.2. *(cf. Theorem 1 in [Mey76]) Let $\mathcal{X}$ be a Polish space and $D([0,T];\mathcal{X})$ be the space of $\mathcal{X}$-valued càdlàg trajectories with the Skorokhod $\mathcal{J}_1$-topology. If $\mathfrak{A}$ is a random variable taking values in $D([0,T];\mathcal{X})$, then there exists a family of $\mathcal{B}([0,T])\times\mathcal{F}$-measurable non-negative measures $E^t(dU)$, $(\omega,t) \in \Omega\times[0,T]$, on $D([0,T];\mathcal{X})$ and a random-variable ζ satisfying $\mathbf{P}(\zeta < T) = 0$ such that $E^t(D([0,T];\mathcal{X})) = 1$ for $t < \zeta$ and $E^t(D([0,T];\mathcal{X})) = 0$ for $t \geq \zeta$. In addition, E^t is càdlàg in the topology of weak convergence, $E^t = E^{t+}$ for all $t \in [0,T]$, and for each continuous and bounded functional F on $D([0,T];\mathcal{X})$, the process $E^t(F)$ is the càdlàg version of $\mathbf{E}[F(\mathfrak{A})|\mathcal{F}_t]$. If $G : \Omega\times$*

$[0,T] \times [0,T] \times D([0,T];\mathcal{X}) \to \mathbf{R}^{d_2}$ is bounded and $\mathcal{O} \times \mathcal{B}([0,T]) \times \mathcal{B}(D([0,T];\mathcal{X}))$-measurable, then

$$\int_{D([0,T];\mathcal{X})} G_t(\omega, t, U) E^t(dU) = E^t(G_t)$$

is the optional projection of $G_t(\mathfrak{U}) = G_t(\omega, t, \mathfrak{U})$. Furthermore, if $G = G_t(\omega, t, U)$ is bounded and $\mathcal{P} \times \mathcal{B}([0,T]) \times \mathcal{B}(D([0,T];\mathcal{X}))$-measurable, then $E^{t-}(G_t)$ is the predictable projection of $G_t(\mathfrak{U}) = G_t(\omega, t, \mathfrak{U})$.

Proof. We follow the proof of Theorem 1 in [Mey76]. Since $D([0,T];\mathcal{X})$ is a Polish space, for each $t \in [0,T]$, there is family of probability measures $\tilde{E}^t_\omega(dw)$, $\omega \in \Omega$, on $D([0,T];\mathcal{X})$ such that for each $A \in \mathcal{B}(D([0,T];\mathcal{X}))$, $\tilde{E}^t(A)$ is $\mathcal{F}_t$-measurable and $\mathbf{P}$-a.s.

$$\mathbf{P}(\mathfrak{U} \in A | \mathcal{F}_t) = \tilde{E}^t(A).$$

For each $\omega \in \Omega$, let $I(\omega)$ be the set of all $t \in (0,T]$ such that for any bounded continuous function F on $D(([0,T];\mathcal{X})$, the function

$$r \mapsto \tilde{E}^r_\omega(F) = \int_{D([0,T];\mathcal{X})} F(w) \tilde{E}^r(dw)$$

has a right-hand limit on $[0,s) \cap \mathbf{Q}$ and a left-hand limit on $(0,s] \cap \mathbf{Q}$ for every rational $s \in [0,T] \cap \mathbf{Q}$. Let $\zeta(\omega) = \sup(t : t \in I(\omega)) \wedge T$. It is easy to see that $\mathbf{P}(\xi < T) = 0$. We set $\tilde{E}^t_\omega = 0$ if $\xi(\omega) < t \le T$. The function $\tilde{E}^t_\omega$ has left-hand and right-hand limits for all $t \in \mathbf{Q} \cap [0,T]$. We define $E^t_\omega = \tilde{E}^{t+}_\omega$ for each $t \in [0,T)$ (the limit is taken along the rationals), and E^T_ω is the left-hand limit at T along the rationals. The statement follows by repeating the proof of Theorem 1 in [Mey76] in an obvious way. □

3.4.3 Estimates of Hölder continuous functions

In the coming lemmas, we establish some properties of weighted Hölder spaces that are used Section 3.3.5 and the proof of Theorem 3.2.5.

Lemma 3.4.3. *Let $\beta \in (0,1]$ and $\theta_1, \theta_2 \in \mathbf{R}$ with $\theta_1 - \theta_2 \le \beta$.*

(1) There is a constant $c_1 = c_1(\theta_2, \beta)$ such that for all $\phi : \mathbf{R}^{d_1} \to \mathbf{R}$ with $|r_1^{-\theta_1}\phi|_0 + [r_1^{-\theta_2}\phi]_\beta =: N_1 < \infty$,

$$|\phi(x) - \phi(y)| \le c_1 N_1 (r_1(x)^{\theta_2} \vee r_1(y)^{\theta_2}) |x-y|^\beta,$$

for all $x, y \in \mathbf{R}^{d_1}$.

(2) *Conversely, if* $\phi : \mathbf{R}^{d_1} \to \mathbf{R}$ *satisfies* $|r_1^{-\theta_1}\phi|_0 < \infty$ *and there is a constant* N_2 *such that for all* $x, y \in \mathbf{R}^{d_1}$,

$$|\phi(x) - \phi(y)| \leq N_2(r_1(x)^{\theta_2} \vee r_1(y)^{\theta_2})|x - y|^\beta,$$

then

$$[r_1^{-\theta_2}\phi]_\beta \leq c_1|r_1^{-\theta_1}\phi|_0 + N_2.$$

Proof. (1) For all x, y with $r_1(x)^{\theta_2} \geq r_1(y)^{\theta_2}$, we have

$$\begin{aligned}|\phi(x) - \phi(y)| &\leq r_1(x)^{\theta_2}[r_1^{-\theta_2}\phi]_\beta|x - y|^\beta + r_1(y)^{\theta_1-\theta_2}|r_1^{-\theta_1}\phi|_0|r_1^{\theta_2}(x) - r_1(y)^{\theta_2}| \\ &\leq ([r_1^{-\theta_2}\phi]_\beta + c_1|r_1^{-\theta}\phi|_0)r_1(x)^{\theta_2}|x - y|^\beta,\end{aligned}$$

where $c_1 := 1 + \sup_{t\in(0,1)} \frac{1-t^{\theta_2}}{(1-t)^\beta}$ if $\theta_2 \geq 0$ and $c_1 := 1 + \sup_{t\in(1,\infty)} \frac{(t^{\theta_2}-1)t^\beta}{(t-1)^\beta}$ if $\theta_2 < 0$, which proves the first claim. (2) For all x and y with $r_1(x)^{\theta_2} > r_1(y)^{\theta_2}$, we have

$$\begin{aligned}&|r_1(x)^{-\theta_2}\phi(x) - r_1(y)^{-\theta_2}\phi(y)| \\ &\leq r_1(x)^{-\theta_2}|\phi(x) - \phi(y)| + r_1(y)^{\theta_1-\theta_2}|r_1^{-\theta_1}(y)\phi(y)||r_1(y)^{\theta_2}r_1(x)^{-\theta_2} - 1| \\ &\leq (c_1|r^{-\theta_1}\phi|_0 + N_2)|x - y|^\beta,\end{aligned}$$

which proves the second claim. □

Lemma 3.4.4. *Let* $\beta, \mu \in (0, 1]$ *and* $\theta_1, \theta_2, \theta_3, \theta_4 \in \mathbf{R}$ *with* $\theta_1 - \theta_2 \leq \beta$, $\theta_3 - \theta_4 \leq \mu$, *and* $\theta_3 \geq 0$. *If* $\phi : \mathbf{R}^{d_1} \to \mathbf{R}$ *and* $H : \mathbf{R}^{d_1} \to \mathbf{R}^{d_1}$ *are such that*

$$|r_1^{-\theta_1}\phi|_0 + [r_1^{-\theta_2}\phi]_\beta =: N_1 < \infty \quad \textit{and} \quad |r_1^{-\theta_3}H|_0 + [r_1^{-\theta_4}H]_\mu =: N_2 < \infty,$$

then

$$|\phi \circ H \cdot r_1^{-\theta_1\theta_3}|_0 \leq |r_1^{-\theta_1}\phi|_0(1 + |r_1^{-\theta_3}H|_0) \leq N_1(1 + N_2)^{\theta_1}$$

and there is a constant $N = N(\beta, \mu, \theta_1, \theta_2)$ *such that*

$$[\phi \circ H \cdot r_1^{-\theta_2\theta_3-\beta\theta_4}]_{\beta\mu} \leq NN_1(1 + N_2)^{\theta_2+\beta}.$$

Proof. For all x, we have

$$r_1(H(x)) \leq (1 + |r_1^{-\theta_3}H|_0)r_1(x)^{\theta_3} \leq (1 + N_2)r_1(x)^{\theta_3},$$

and hence

$$|\phi \circ H \cdot r_1^{-\theta_1\theta_3}|_0 \leq |r_1^{-\theta_1}\phi|_0|r_1^{\theta_1} \circ H \cdot r_1^{-\theta_1\theta_3}|_0 \leq N_1(1 + N_2)^{\theta_1}.$$

Using Lemma 3.4.3, for all x and y, we get

$$\begin{aligned}|\phi(H(x)) - \phi(H(y))| &\leq NN_1(r_1(H(x)) \vee r_1(H(y)))^{\theta_2}|H(x) - H(y)|^\beta \\ &\leq NN_1(1+N_2)^{\theta_2}(r_1(x) \vee r_1(y))^{\theta_2\theta_3}N_2^\beta(r_1(x) \vee r_1(y))^{\beta\theta_4}|x-y|^{\beta\mu} \\ &\leq NN_1(1+N_2)^{\theta_2+\beta}(r_1(x) \vee r_1(y))^{\theta_2\theta_3+\beta\theta_4}|x-y|^{\beta\mu},\end{aligned}$$

for some constant $N = N(\beta, \mu, \theta_1, \theta_2)$. Noting that

$$\theta_1\theta_3 - \theta_2\theta_3 - \beta\theta_4 = (\theta_1 - \theta_2)\theta_3 - \beta\theta_4 \leq \beta(\theta_3 - \theta_4) \leq \beta\mu,$$

we apply Lemma 3.4.3 to complete the proof. □

Remark 3.4.5. Let $\beta \in (0,1]$ and $\theta_1, \theta_2 \in \mathbf{R}$. Then there is a constant $N = N(\beta, \theta_1, \theta_2)$ such that for all $\phi : \mathbf{R}^{d_1} \to \mathbf{R}$ with $|r_1^{-\theta_1}\phi|_0 + [r_1^{-\theta_2}\phi]_\beta =: N_1 < \infty$, we have $|r^{-\theta}\phi|_\beta \leq NN_1$, where $\theta = \max\{\theta_1, \theta_2\}$. In particular, if in Lemma 3.4.4, $\theta_1 = \theta_2$ and $\theta_4 \geq 0$, then

$$|\phi \circ H \cdot r^{-\theta_1\theta_3 - \beta\theta_4}|_{\beta\mu} \leq NN_1(1+N_2)^{\theta_1+\beta}.$$

Proof. If $\theta_2 \geq \theta_1$, then the claim is obvious and if $\theta_1 > \theta_2$, for all x and y, we find

$$\begin{aligned}|r_1(x)^{-\theta_1}\phi(x) - r_1(y)^{-\theta_1}\phi(y)| &\leq r_1(x)^{\theta_2-\theta_1}|r_1(x)^{-\theta_2}\phi(x) - r_1(y)^{-\theta_2}\phi(y)| \\ &\quad + \left|\frac{r(y)^{\theta_1-\theta_2}}{r(x)^{\theta_1-\theta_2}} - 1\right| |r_1^{-\theta_1}\phi|_0 \leq N_1(1+c_1)|x-y|^\beta,\end{aligned}$$

where $c_1 := \sup_{t\in(0,1)} \frac{1-t^{\theta_1-\theta_2}}{(1-t)^\beta}$. □

Lemma 3.4.6. *For any $\theta \geq 0$ and $\beta > 1$, there are constants $N_1 = N_1(d_1, \theta, \beta)$ and $N_2(d_1, \theta, \beta)$ such that for all $\phi : \mathbf{R}^{d_1} \to \mathbf{R}$,*

$$N_1|r_1^{-\theta}\phi|_\beta \leq \sum_{|\gamma|\leq[\beta]^-} |r_1^{-\theta}\partial^\gamma\phi|_0 + \sum_{|\gamma|=[\beta]^-} |r_1^{-\theta}\partial^\gamma\phi|_{\{\beta\}^+} \leq N_2|r_1^{-\theta}\phi|_\beta. \tag{3.4.1}$$

Proof. For any multi-index γ with $|\gamma| \leq [\beta]^-$ and x, we have

$$\partial^\gamma(r_1^{-\theta}\phi)(x) = \sum_{\substack{\gamma_1+\gamma_2+=\gamma \\ |\gamma_1|\geq 1}} r_1(x)^\theta\partial^{\gamma_1}(r_1^{-\theta})(x)r_1(x)^{-\theta}\partial^{\gamma_2}\phi(x) + r_1(x)^{-\theta}\partial^\gamma\phi(x).$$

It is easy to show by induction that for all multi-indices γ, $|r_1^\theta\partial^\gamma(r_1^{-\theta})|_1 < \infty$. Moreover, for all multi-indices γ with $|\gamma| < [\beta]^-$,

$$|r_1^{-\theta}\partial^\gamma\phi|_1 \leq |\nabla(r_1^{-\theta}\partial^\gamma\phi)| \leq |r_1^{-\theta}\nabla(r_1^{-\theta})|_0|r_1^{-\theta}\partial^\gamma\nabla\phi|_0.$$

Thus, for any multi-index γ with $|\gamma| \le [\beta]^-$,

$$|\partial^\gamma (r_1^{-\theta}\phi)|_0 \le \sum_{\substack{\gamma_1+\gamma_2+=\gamma \\ |\gamma_1|\ge 1}} |r_1^\theta \partial^{\gamma_1}(r_1^{-\theta})|_0 |r_1^{-\theta}\partial^{\gamma_2}\phi|_0 + |r_1^{-\theta}\partial^\gamma \phi|_0$$

and for any multi-index γ with $|\gamma| = [\beta]^-$,

$$|\partial^\gamma (r_1^{-\theta}\phi)|_{\{\beta\}^+} \le \sum_{\substack{\gamma_1+\gamma_2+=\gamma \\ |\gamma_1|\ge 1}} |r_1^\theta \partial^{\gamma_1}(r_1^{-\theta})|_1 |r_1^{-\theta}\nabla(r_1^{-\theta})|_0 |r_1^{-\theta}\partial^{\gamma_2}\nabla\phi|_0 + |r_1^{-\theta}\partial^\gamma \phi|_0.$$

This proves the leftmost inequality in (3.4.1). For all $i \in \{1,\dots,d\}$ and x,

$$r_1^{-\theta}\partial_i\phi(x) = \partial_i(r_1^{-\theta}\phi)(x) - r_1(x)^{-\theta}\phi(x) r_1(x)^\theta \partial_i(r_1^{-\theta})(x).$$

It follows by induction that for all multi-indices γ with $|\gamma| \le [\beta]^-$ and x, $r_1^{-\theta}\partial^\gamma\phi(x)$ is a sum of $\partial^\gamma(r_1^{-\theta}\phi)(x)$, a finite sum of terms, each of which is a product of one term of the form $\partial^{\tilde\gamma}(r_1^{-\theta}\phi)(x)$, $|\tilde\gamma| < |\gamma|$, and a finite number of terms of the form $\partial^{\gamma_1}(r_1^\theta)\partial^{\gamma_2}(r_1^{-\theta})$, $|\gamma_1|, |\gamma_2| \le |\gamma|$. Since for all multi-indices γ_1 and γ_2, we have $|\partial^{\gamma_1}(r_1^\theta)\partial^{\gamma_2}(r_1^{-\theta})|_1 < \infty$, the rightmost inequality in (3.4.1) follows. □

Corollary 3.4.7. *For any $\theta \ge 0$ and $\beta > 1$, there are constants $N_1 = N_1(d_1,\theta,\beta)$ and $N_2(d_1,\theta,\beta)$ such that for all $\phi : \mathbf{R}^{d_1} \to \mathbf{R}$,*

$$N_1|r_1^{-\theta}\phi|_\beta \le |r_1^{-\theta}\phi|_0 + \sum_{|\gamma|=[\beta]^-} |r_1^{-\theta}\partial^\gamma\phi|_{\{\beta\}^+} \le N_2 |r_1^{-\theta}\phi|_\beta.$$

Proof. It is well known that for an arbitrary unit ball $B \subset \mathbf{R}^{d_1}$ and any $1 \le k < [\beta]^-$, there is a constant N such that for any $\varepsilon > 0$,

$$\sup_{x\in B, |\gamma|=k} |\partial^\gamma\phi| \le N(\varepsilon \sup_{x\in B,|\gamma|=[\beta]^-} |\partial^\gamma\phi(x)| + \varepsilon^{-k}\sup_{x\in B}|\phi(x)|).$$

Let $U_0 = \{x \in \mathbf{R}^{d_1} : |x| \le 1\}$ and $U_j = \{x \in \mathbf{R}^{d_1} : 2^{j-1} \le |x| \le 2^j\}$, $j \ge 1$. For all j, we have

$$\begin{aligned}
\sup_{x\in U_j,|\gamma|=k} |\partial^\gamma\phi(x)| &= \sup_{B\subseteq U_j}\sup_{x\in B,|\gamma|=k} |\partial^\gamma\phi(x)| \\
&\le N(\varepsilon \sup_{B\subseteq U_j}\sup_{x\in B,|\gamma|=[\beta]^-} |\partial^\gamma\phi(x)| + \varepsilon^{-k}\sup_{B\subseteq U_j}\sup_{x\in B}|\phi(x)|) \\
&\le N(\varepsilon \sup_{x\in U_j,|\gamma|=[\beta]^-} |\partial^\gamma\phi(x)| + \varepsilon^{-k}\sup_{x\in U_j}|\phi(x)|).
\end{aligned}$$

Since for every j,

$$2^{-\theta/2}2^{-j\theta}\sup_{x\in U_j,|\gamma|=k}|\partial^\gamma\phi(x)|\le\sup_{x\in U_j,|\gamma|=k}|r^{-\theta}\partial^\gamma\phi(x)|\le 2^{\theta}2^{-(j-1)\theta}\sup_{x\in U_j,|\gamma|=k}|\partial^\gamma\phi(x)|,$$

we see that

$$\begin{aligned}2^{-\theta/2}\sup_j 2^{-j\theta}\sup_{x\in U_j,|\gamma|=k}|\partial^\gamma\phi(x)|&\le\sup_j\sup_{x\in U_j,|\gamma|=k}|r^{-\theta}\partial^\gamma\phi(x)|=|r^{-\theta}\partial^\gamma\phi|_0\\&\le 2^{\theta}\sup_j 2^{-j\theta}\sup_{x\in U_j,|\gamma|=k}|\partial^\gamma\phi(x)|,\end{aligned}$$

and the statement follows. □

Remark 3.4.8. If $\phi:\mathbf{R}^{d_1}\to\mathbf{R}$ is such that $|r^{-\theta_1}\phi|_0+|r^{-\theta_2}\nabla\phi|_0<\infty$ for $\theta_1,\theta_2\in\mathbf{R}$ with $\theta_1-\theta_2\le 1$, then

$$[r^{-\theta_2}\phi]_1\le N(|r^{-\theta_1}\phi|_0+|r^{-\theta_2}\nabla\phi|_0)$$

Proof. Indeed, for all x and y, we have

$$\begin{aligned}|\phi(x)-\phi(y)|&\le|r^{-\theta_2}\nabla\phi|_0\int_0^1 r^{\theta_2}(x+s(y-x))ds|y-x|\\&\le|r^{-\theta_2}\nabla\phi|_0(r(y)^{\theta_2}\vee r(x)^{\theta_2})|y-x|,\end{aligned}$$

and hence the claim follows from Lemma 3.4.3. □

Lemma 3.4.9. *Let $n\in\mathbf{N}$, $\beta,\mu\in(0,1]$, $\theta_3,\theta_4\ge 0$ be such that $\theta_3-\theta_4\le 1$. There is a constant $N=N(d_1,\theta_1,\theta_3,\theta_4,n,\beta)$ such that for all $\phi:\mathbf{R}^{d_1}\to\mathbf{R}$ with $r_1^{-\theta_1}\phi\in C^{n+\beta}(\mathbf{R}^{d_1},\mathbf{R}^{d_1})$ and $H:\mathbf{R}^{d_1}\to\mathbf{R}^{d_1}$ with*

$$|r_1^{-\theta_3}H|_0+|r_1^{-\theta_4}\nabla H|_{n-1+\mu}=:N_2<\infty,$$

we have

$$|\phi\circ H\cdot r^{-\theta_1\theta_3}|_0\le|r_1^{-\theta_1}\phi|_0(1+|r_1^{-\theta_3}H|_0)^{\theta_1}$$

and

$$|r_1^{-\theta_1\theta_3-\theta_4(n+\mu\wedge\beta)}\nabla(\phi\circ H)|_{n-1+\mu\wedge\beta}\le N|r_1^{-\theta_1}\phi|_{n+\beta}(1+N_2)^{\theta_1+\mu\wedge\beta+n}.$$

Proof. It follows immediately from Lemma 3.4.4 and Remark 3.4.8 that

$$|\phi\circ H\cdot r^{-\theta_1\theta_3}|_0\le|r_1^{-\theta_1}\phi|_0(1+|r_1^{-\theta_3}H|_0)^{\theta_1}.$$

Using induction, we get that for all x and $|\gamma|=n$,

$$\partial^\gamma(\phi(H(x)))=\mathcal{I}_1^\gamma(x)+\mathcal{I}_2^\gamma(x)+I_3^\gamma(x),$$

where

$$\mathcal{I}_1^{\gamma}(x)=\sum_{i=1}^{d_1}\partial_i\phi(H(x))\partial^{\gamma}H^i(x)$$

$\mathcal{I}_2^{\gamma}(x)$ is a finite sum of terms of the form

$$\partial_{i_1}\cdots\partial_{i_{|\gamma|}}\phi(H(x))\partial^{\tilde{\gamma}_1}H^{i_1}\cdots\partial^{\tilde{\gamma}_{|\gamma|}}H^{i_{|\gamma|}}$$

with $i_1,\ldots,i_{|\gamma|}\in\{1,2,\ldots,d\}$, $|\tilde{\gamma}_1|=\cdots=|\tilde{\gamma}_{|\gamma|}|=1$, and $\sum_{k=1}^{|\gamma|}\tilde{\gamma}_k=\gamma$, if $n\geq 2$ and zero otherwise, and where $\mathcal{I}_3^{\gamma}(x)$ is a finite sum of terms of the form

$$\partial_{i_1}\cdots\partial_{i_k}\phi(H(x))\partial^{\tilde{\gamma}_1}H^{i_1}(x)\cdots\partial^{\tilde{\gamma}_k}H^{i_k}(x)$$

with $2\leq k<n$, $i_1,i_2,\ldots,i_k\in\{1,\ldots,d\}$, and $\sum_{j=1}^{k}\tilde{\gamma}_j=\gamma$, $1\leq|\tilde{\gamma}_j|<|\gamma|$, if $n\geq 3$, and zero otherwise. Thus, owing to Lemmas 3.4.4 and 3.4.6, for any multi-index γ with $|\gamma|=n$, we have

$$|r_1^{-\theta_3\theta_1-\theta_4}\mathcal{I}_1^{\gamma}|_0\leq N|r_1^{-\theta_1}\nabla\phi|_0(1+|r_1^{-\theta_3}H|_0)^{\theta_1}|r_1^{-\theta_4}\partial^{\gamma}H|_0,$$
$$|r_1^{-\theta_3\theta_1-n\theta_4}\mathcal{I}_2^{\gamma}|_0\leq N|r_1^{-\theta_1}\partial^{\gamma}\phi|_0(1+|r_1^{-\theta_3}H|_0)^{\theta_1}|r_1^{-\theta_4}\nabla H|_0^{n},$$

and

$$|r_1^{-\theta_3\theta_1-(n-1)\theta_4}\mathcal{I}_3^{\gamma}|_0\leq N|r_1^{-\theta_1}\phi|_{n-1}(1+|r_1^{-\theta_3}H|_0+|r_1^{-\theta_4}\nabla H|)^{\theta_1+n-1},$$

and hence

$$|r^{-\theta_1\theta_3-n\theta_4}\partial^{\gamma}(\phi\circ H)|_0\leq N|r_1^{-\theta_1}\phi|_n(1+|r_1^{-\theta_3}H|_0+|r_1^{-\theta_4}\nabla H|)^{\theta_1+n}.$$

Appealing again to Lemmas 3.4.4 and 3.4.6, for all multi-indices γ with $|\gamma|=n$, we get

$$|r_1^{-\theta_1\theta_3-(1+\mu\wedge\beta)\theta_4}\mathcal{I}_1^{\gamma}|_{\mu\wedge\beta}\leq N|r_1^{-\theta_1}\phi|_{1+\mu\wedge\beta}\,(1+N_2)^{\theta_1+\mu\wedge\beta+1}\,,$$

$$|r_1^{-\theta_1\theta_3-(n+\mu\wedge\beta)\theta_4}\mathcal{I}_2^{\gamma}|_{\mu\wedge\beta}+|r_1^{-\theta_1\theta_3-(n-1+\mu\wedge\beta)\theta_4}\mathcal{I}_3^{\gamma}|_{\mu\wedge\beta}\leq N|r_1^{-\theta_1}\phi|_{n+\mu\wedge\beta}\,(1+N_2)^{\theta_1+n+\mu\wedge\beta}\,.$$

Then applying Lemmas 3.4.4 and 3.4.6, we complete the proof. □

We shall now provide some useful estimates of composite functions of diffeomorphisms.

Lemma 3.4.10. *Let $H:\mathbf{R}^{d_1}\to\mathbf{R}^{d_1}$ be continuously differentiable and assume that for all $x\in\mathbf{R}^{d_1}$,*

$$|H(x)|\leq L_0+L_1|x|\quad\textit{and}\quad|\nabla H(x)|\leq L_2.$$

Assume that for all $x\in\mathbf{R}^{d_1}$, $\kappa(x)=(I_{d_1}+\nabla H(x))^{-1}$ exists and $|\kappa(x)|\leq N_\kappa$.

(1) Then the mapping $\tilde{H}(x):=x+H(x)$ is a diffeomorphism with $\tilde{H}^{-1}(x)=x-H(\tilde{H}^{-1}(x))$

=: $x + F(x)$ *and for all* $x \in \mathbf{R}^{d_1}$,

$$|F(x)| \le L_0 + L_1 L_0 N_\kappa + L_1 N_\kappa |x|, \quad |\nabla F(x)| \le N_\kappa L_2, \qquad |\left(I_{d_1} + \nabla F(x)\right)^{-1}| \le 1 + L_2.$$

For all $p \in \mathbf{R}$, *there is a constant* $N = N(L_0, L_1, N_\kappa, p)$ *such that for all* $x \in \mathbf{R}^{d_1}$,

$$\frac{r_1^p(\tilde{H}(x))}{r_1^p(x)} + \frac{r_1^p(\tilde{H}^{-1}(x))}{r_1^p(x)} \le N, \quad r_1^{-1}(x)|H^i(x) + F^{k;i}(x)| \le N[H]_1 |r_1^{-1} H|_0.$$

Moreover, there is a constant $N = N(L_0, L_1, N_\kappa, p)$ *such that*

$$\left|\frac{r_1^p(\tilde{H})}{r_1^p} - 1 + \mathbf{1}_{(1,2]}(\alpha) p H^i r_1^{-2} x^i\right|_\alpha + \left|\frac{r_1^p(\tilde{H}^{-1})}{r_1^p} - 1 - \mathbf{1}_{(1,2]}(\alpha) p F^i r_1^{-2} x^i\right|_\alpha$$
$$\le N(|r_1^{-1} H|_0^{[\alpha]^- + 1} + [H]_1^{[\alpha]^- + 1}).$$

(2) *If for some* $\beta > 1$, $|\nabla H|_{\beta - 1} \le L_3$, *then there is a constant* $N = N(d_1, \beta, N_\kappa, L_3)$ *such that*

$$|\nabla F|_{\beta-1} \le N |\nabla H|_{\beta - 1}. \tag{3.4.2}$$

(3) *If for some* $\beta \ge 1$, $|\nabla H|_{\beta-1} \le L_3$, *then for all* $\theta \ge 0$, *there is a constant* $N = N(d_1, \beta, N_\kappa, L_1, L_3, \theta)$ *such that*

$$\left|\frac{r_1^\theta \circ \tilde{H}^{-1}}{r_1^\theta} - 1\right|_\beta \le N(|r_1^{-1} H|_0 + |\nabla H|_{\beta-1}).$$

(4) *If* $|H|_0 \le L_4$, *and for some* $\beta > 0$, $|\nabla H|_{\beta \vee 1 - 1} \le L_5$ *and* $\phi : \mathbf{R}^{d_1} \to \mathbf{R}$ *is such that for some* $\mu \in (0, 1]$ *and* $\theta \ge 0$, $r_1^{-\theta}\phi \in C^{\beta+\mu}(\mathbf{R}^{d_1}; \mathbf{R})$, *then there is a constant* $N = N(d_1, \beta, \mu, N_\kappa, L_4, L_5, \theta)$ *such that*

$$\begin{aligned} |r_1^{-\theta}(\phi \circ \tilde{H}^{-1} - \phi)|_\beta &\le N |r_1^{-\theta}\phi|_\beta (|H|_0 + |\nabla H|_{\beta \vee 1 - 1}) \\ &+ N \mathbf{1}_{(0,1]}(\{\beta\}^+ + \mu) \sum_{|\gamma| = [\beta]^-} [\partial^\gamma (r_1^{-\theta}\phi)]_{\{\beta\}^+ + \mu} L_4^\mu \\ &+ N \mathbf{1}_{(1,2]}(\{\beta\}^+ + \mu) \sum_{|\gamma| = [\beta]^-} \left([\nabla \partial^\gamma (r_1^{-\theta}\phi)]_{\{\beta\}^+ + \mu - 1} L_4^\mu + |\nabla \partial^\gamma (r_1^{-\theta}\phi)|_0 |\nabla H|_0\right). \end{aligned}$$

Proof. (1) Since $(I_{d_1} + \nabla H(x))^{-1}$ exists for all x, it follows from Theorem 0.2 in [DMGZ94] that the mapping $\tilde{H}$ is a global diffeomorphism. For all x, we easily verify $\tilde{H}^{-1}(x) = x - H(\tilde{H}^{-1}(x))$ by substituting $\tilde{H}(x)$ into the expression. Simple computations show that for all x, we have

$$|\nabla \tilde{H}(x)| \le 1 + L_2, \ |\nabla \tilde{H}^{-1}(x)| = |\kappa(\tilde{H}^{-1}(x))| \le N_\kappa,$$

$$|\nabla F(x)| = |\nabla H(\tilde{H}^{-1}(x))\nabla \tilde{H}^{-1}(x)| \leq N_\kappa L_2,$$
$$|(I_{d_1} + \nabla F(x))^{-1}| = |\nabla \tilde{H}^{-1}(x)^{-1}| = |\kappa(\tilde{H}^{-1}(x))^{-1}| = |I_{d_1} + \nabla H(\tilde{H}^{-1}(x))| \leq 1 + L_2.$$

For all x and y, we easily obtain

$$|\tilde{H}(x) - \tilde{H}(y)| \leq (1 + L_2)|x - y|, \quad |\tilde{H}^{-1}(x) - \tilde{H}^{-1}(y)| \leq N_\kappa |x - y|,$$

and hence

$$N_\kappa^{-1}|x - y| \leq |\tilde{H}(x) - \tilde{H}(y)|, \quad (1 + L_2)^{-1}|x - y| \leq |\tilde{H}^{-1}(x) - \tilde{H}^{-1}(y)|. \tag{3.4.3}$$

Making use of (3.4.3), for all x, we get

$$N_\kappa^{-1}|x| \leq L_0 + |\tilde{H}(x)|, \quad |\tilde{H}^{-1}(x)| \leq N_\kappa L_0 + N_\kappa |x|, \quad |x| \leq L_0 + L_1|\tilde{H}^{-1}(x)|,$$

and thus

$$|F(x)| \leq L_0 + L_1 N_\kappa L_0 + L_1 N_\kappa |x|.$$

The rest of the estimates then follow easily from the above estimates and Taylor's theorem.

(2) Using the chain rule, for all x, we obtain

$$\nabla F(x) = -\nabla H(\tilde{H}^{-1}(x))\nabla \tilde{H}^{-1}(x) = -\nabla H(\tilde{H}^{-1}(x))\kappa(\tilde{H}^{-1}(x)), \tag{3.4.4}$$

and hence $|\nabla F|_0 \leq N_\kappa |\nabla H|_0$. For all x and y, we have

$$\kappa(\tilde{H}^{-1}(y)) - \kappa(\tilde{H}^{-1}(x)) = \kappa(y)[\nabla H(\tilde{H}^{-1}(x)) - \nabla H(\tilde{H}^{-1}(y))]\kappa(x),$$

and thus since $[\tilde{H}^{-1}]_1 \leq (1 + N_\kappa L_3)$ by part (1), we have for all $\delta \in (0, 1 \wedge \beta]$,

$$[\kappa(\tilde{H}^{-1})]_\delta \leq N_\kappa^2 (1 + N_\kappa L_3)^\delta [\nabla H]_\delta.$$

It follows that there is a constant $N = N(N_\kappa, L_3)$ such that for all $\delta \in (0, 1 \wedge \beta]$,

$$|\nabla F|_\delta \leq N|H|_\delta.$$

It is well-known that the inverse map $\mathfrak{I}$ on the set of invertible $d_1 \times d_1$ matrices is infinitely differentiable and for each n, there exists a constant $N = N(n, d_1)$ such that for all invertible matrices A, the n-th derivative of $\mathfrak{I}$ evaluated at A, denoted $\mathfrak{I}^{(n)}(A)$, satisfies

$$|\mathfrak{I}^{(n)}(A)| \leq N|A^{-n-1}| \leq N|A^{-1}|^{n+1}.$$

Using induction we find that for all multi-indices γ with $|\gamma| \le [\beta]^-$ and for all x, $\partial^\gamma F(x)$ is a finite sum of terms, each of which is a finite product of

$$\partial^{\bar\gamma} H(\tilde H^{-1}(x)), \quad \kappa(\tilde H^{-1}(x))^{\bar n}, \quad \Im^{(\bar n-1)}(I + \nabla H(\tilde H^{-1}(x))), \quad |\bar\gamma| \le |\gamma|, \quad \bar n \in \{1, \dots, |\gamma|\}.$$

Therefore, differentiating (3.4.4) and estimating directly we easily obtain (3.4.2).
(3) For each x, we have

$$\begin{aligned}\frac{r_1(\tilde H^{-1}(x))^\theta}{r_1(x)^\theta} - 1 &= r_1(x)^{-\theta}\int_0^1 r_1(G_s(x))^{\theta-2} G_s(x)^* F(x)ds\\ &= \int_0^1 \frac{r_1^{\theta-1}(G_s(x))}{r_1(x)^{\theta-1}} K(G_s(x))^* ds r_1(x)^{-1}F(x),\end{aligned}$$

where $G_s(x) := x + sF(x)$, $s \in [0,1]$, and $J(x) := r_1(x)^{-1}x$. According to part (1) and (2), we have $|r_1^{-1}F|_0 \le N|r_1^{-1}H|_0$ and $|\nabla F|_{\beta-1} \le N|\nabla H|_{\beta-1}$, and hence

$$|r_1^{-1}G_s|_0 \le N(1 + |r_1^{-1}H|_0), \quad |\nabla G_s(x)|_{\beta-1} \le N(1 + |\nabla H|_{\beta-1}).$$

and

$$|J \circ G_s|_\beta \le N(1 + |r_1^{-1}H|_0 + |\nabla H|_{\beta-1}),$$

for some constant N independent of s. Moreover, using Lemma 3.4.9 we find

$$|r_1^{\theta-1} \circ G_s \cdot r_1^{1-\theta}|_\beta \le N\left(1 + |r_1^{-1}H|_0 + |\nabla H|_{\beta-1}\right)^{\theta+\beta}.$$

The statement then follows.
(4) First, we will consider the case $\theta = 0$. By part (1), we have that for all $\bar\mu \in (0, (\beta+\mu)\wedge 1]$,

$$|\phi \circ \tilde H^{-1} - \phi|_0 \le [\phi]_{\bar\mu}|H \circ \tilde H^{-1}|_0^{\mu} \le [\phi]_{\bar\mu}|H|_0^{\bar\mu}.$$

First, let us consider the case $\beta \le 1$. For each x, let $\mathcal{J}(x) = \phi(\tilde H^{-1}(x)) - \phi(x)$. For all x and y, it is clear that

$$|\mathcal{J}(x) - \mathcal{J}(y)| \le A(x,y) + B(x,y) + C(x,y),$$

where

$$A(x,y) := |\mathcal{J}(x)|\mathbf{1}_{[L_4,\infty)}(|x-y|), \quad B(x,y) := |\mathcal{J}(y)|\mathbf{1}_{[L_4,\infty)}(|x-y|),$$

and

$$C(x,y) := |\mathcal{J}(x) - \mathcal{J}(y)|\mathbf{1}_{[0,L_4)}(|x-y|).$$

Moreover, owing to part (1), if $\beta + \mu \le 1$, then for all x and y, we have

$$A(x,y) \le [\phi]_{\beta+\mu}L_4^{\beta+\mu}\mathbf{1}_{[L_4,\infty)}(|x-y|) \le [\phi]_{\beta+\mu}L_4^{\mu}|x-y|^{\{\beta\}^+},$$

$$B(x,y) \le [\phi]_{\beta+\mu} L_4^{\mu} |x-y|^{\beta},$$

and

$$\begin{aligned} C(x,y) &\le [\phi]_{\beta+\mu} [[\tilde{H}^{-1}]_1^{\beta+\mu} |x-y|^{\beta+\mu} \mathbf{1}_{[0,L_4)}(|x-y|) + [\phi]_{\beta+\mu} |x-y|^{\beta+\mu} \mathbf{1}_{[0,L_4)}(|x-y|) \\ &\le N [\phi]_{\beta+\mu} L_4^{\mu} |x-y|^{\beta} \end{aligned}$$

for some constant $N = N(\mu, N_\kappa, L_4)$. Using the identity

$$\begin{aligned} &\mathcal{J}(x) - \mathcal{J}(y) \\ &= -\int_0^1 \left(\nabla\phi\left(x - \theta H(\tilde{H}^{-1}(x))\right) - \nabla\phi\left(y - \theta H(\tilde{H}^{-1}(y))\right)\right) H(\tilde{H}^{-1}(x)) d\theta \\ &\quad - \int_0^1 \nabla\phi\left(y - \theta H(\tilde{H}^{-1}(y))\right) (H(\tilde{H}^{-1}(y)) - H(\tilde{H}^{-1}(x))), \end{aligned}$$

and part (1), if $\beta + \mu > 1$, then there is a constant $N = N(\mu, N_\kappa, L_4)$ such that for all x and y,

$$\begin{aligned} |\mathcal{J}(x) - \mathcal{J}(y)| \mathbf{1}_{[L_4,\infty)}(|x-y|) &\le N([\nabla\phi]_{\beta+\mu-1} |x-y|^{\beta+\mu-1} L_4 \\ &\quad + |\nabla\phi|_0 |x-y| [H]_1) \mathbf{1}_{[L_4,\infty)}(|x-y|) \\ &\le N [\nabla\phi]_{\beta+\mu-1} L_4^{\mu} |x-y|^{\beta} + N |\nabla\phi|_0 |\nabla H|_0 |x-y|. \end{aligned}$$

Moreover, since

$$\begin{aligned} &\mathcal{J}(x) - \mathcal{J}(y) \\ &= \int_0^1 \nabla\phi\left(\tilde{H}^{-1}(x + \theta(y-x))\right)\left(\nabla\tilde{H}^{-1}(x+\theta(y-x)) - I_{d_1}\right)(x-y) d\theta \\ &\quad + \int_0^1 \left(\nabla\phi\left(\tilde{H}^{-1}(x+\theta(y-x))\right) - \nabla\phi(x+\theta(y-x))\right)(x-y) d\theta, \end{aligned}$$

by part (1) and (3.4.2), if $\beta + \mu > 1$, we attain that there is a constant $N = N(\mu, N_\kappa, L_4)$ such that for all x and y,

$$\begin{aligned} |\mathcal{J}(x) - \mathcal{J}(y)| \mathbf{1}_{[0,L_4)}(|x-y|) &\le (|\nabla\phi|_0 |\nabla H|_0 + [\nabla\phi]_{\beta+\mu-1} L_4^{\beta+\mu-1}) |x-y| \mathbf{1}_{[0,L_4)}(|x-y|) \\ &\le |\nabla\phi|_0 |\nabla H|_0 |x-y| + [\nabla\phi]_{\beta+\mu-1} L_4^{\mu} |x-y|^{\beta}. \end{aligned}$$

Combining the above estimates, we get that for all $\beta \le 1$ and $\mu \in (0,1]$, there is a constant $N = N(\mu, N_\kappa, L_4)$ such that

$$[\phi\circ\tilde{H}^{-1} - \phi]_{\beta} \le N \mathbf{1}_{[0,1]}(\beta+\mu) [\phi]_{\beta+\mu} L_4^{\mu} + N \mathbf{1}_{(1,2]}(\beta+\mu)\left([\nabla\phi]_{\beta+\mu-1} + |\nabla\phi|_0 |\nabla H|_0\right). \tag{3.4.5}$$

This proves the desired estimate for $\beta \le 1$ and $\theta = 0$. We now consider the case $\beta > 1$. For $\beta > 1$, it is straightforward to prove by induction that for all multi-indices γ with $1 \le |\gamma| \le [\beta]^-$ and for all x,

$$\partial^\gamma(\phi(\tilde{H}^{-1}))(x) = \mathcal{J}_1^\gamma(x) + \mathcal{J}_2^\gamma(x) + \mathcal{J}_3^\gamma(x) + \mathcal{J}_4^\gamma(x),$$

where

$$\mathcal{J}_1^\gamma(x) := \partial^\gamma\phi(\tilde{H}^{-1}(x)),$$

$$\mathcal{J}_2^\gamma(x) = \partial^\gamma\phi(\tilde{H}^{-1})(\partial_1\tilde{H}^{-1;1})^{\gamma_1}\cdots(\partial_d\tilde{H}^{-1;d})^{\gamma_d} - 1,$$

$\mathcal{J}_3^\gamma(x)$ is a finite sum of terms of the form

$$\partial_{j_1}\cdots\partial_{j_k}\phi(\tilde{H}^{-1}(x))\partial^{\tilde{\gamma}_1}\tilde{H}^{-1;j_1}(x)\cdots\partial^{\tilde{\gamma}_k}\tilde{H}^{-1;j_k}(x)$$

with $1 \le k < [\beta]^-$, $j_1,\dots,j_k \in \{1,\dots,d\}$, and $\sum_{j=1}^k \tilde{\gamma}_j = \gamma$, and $\mathcal{J}_4(x)$ is a finite sum of terms of the form

$$\partial_{j_1}\dots\partial_{j_{[\beta]^-}}\phi(\tilde{H}^{-1}(x))\partial_{i_1}\tilde{H}^{-1;j_1}(x)\cdots\partial_{i_{[\beta]^-}}\tilde{H}^{-1;j_{[\beta]^-}}(x)$$

with $i_1, j_1,\dots,i_{[\beta]^-}, j_{[\beta]^-} \in \{1,\dots,d\}$ and at least one pair $i_k \ne j_k$. Since for all x,

$$\nabla\tilde{H}^{-1}(x) = I + \nabla F(x)$$

and (3.4.2) holds, there is a constant $N = N(d_1,\beta)$ such that

$$\sum_{1\le|\gamma|\le\beta}\sum_{i=2}^4|\mathcal{J}_i^\gamma|_0 + \sum_{|\gamma|=\beta}\sum_{i=2}^4|\mathcal{J}_i^\gamma|_{\{\beta\}^+} \le N|\nabla\phi|_{\beta-1}|\nabla F|_{\beta-1} \le N|\nabla\phi|_{\beta-1}|\nabla H|_{\beta-1}.$$

If $\beta > 2$, then for all multi-indices γ with $1 \le |\gamma| < [\beta]^-$, we get

$$|\mathcal{J}_1^\gamma - \partial^\gamma\phi|_0 = |\partial^\gamma\phi\circ\tilde{H}^{-1} - \partial^\gamma\phi|_0 \le [\partial^\gamma\phi]_1|H|_0.$$

It is easy to see that there is a constant $N = N(L_4, N_\kappa)$ such that for all γ with $|\gamma| = [\beta]^-$ and all $\bar{\mu} \in (0, (\{\beta\}^+ + \mu)\wedge 1]$,

$$|\mathcal{J}_1^\gamma - \partial^\gamma\phi|_0 = |\partial^\gamma\phi\circ\tilde{H}^{-1} - \partial^\gamma\phi|_0 \le [\partial^\gamma\phi]_{\bar{\mu}}|H|_0^{\bar{\mu}}.$$

Moreover, appealing to the estimate (3.4.5) we obtain

$$\begin{aligned}&[\mathcal{J}_1^\gamma - \partial^\gamma\phi]_{\{\beta\}^+}\\ &\le N\mathbf{1}_{[0,1]}(\{\beta\}^+ + \mu)[\partial^\gamma\phi]_{\{\beta\}^+ +\mu}L_4^\mu + N\mathbf{1}_{(1,2]}(\{\beta\}^+ + \mu)\left([\nabla\partial^\gamma\phi]_{\{\beta\}^+ +\mu-1} + |\nabla\partial^\gamma\phi|_0|\nabla H|_0\right).\end{aligned}$$

Let us now consider the case $\theta > 0$. The following decomposition obviously holds for all x:

$$r_1(x)^{-\theta}\phi(\tilde{H}^{-1}(x)) - r_1(x)^{-\theta}\phi(x) = \hat{\phi}(\tilde{H}^{-1}) - \hat{\phi}(x) + \left(\frac{r_1(\tilde{H}^{-1}(x))^{\theta}}{r_1(x)^{\theta}} - 1\right)\hat{\phi}(\tilde{H}^{-1}(x)),$$

where $\hat{\phi} = r_1^{-\theta}\phi \in C^{\beta}(\mathbf{R}^{d_1};\mathbf{R}^{d_1})$. Thus, to complete the proof we require

$$|\hat{\phi}\circ\tilde{H}^{-1}|_{\beta} \le N|\hat{\phi}|_{\beta} \quad \text{and} \quad \left|\frac{r_1^{\theta}\circ\tilde{H}^{-1}}{r_1^{\theta}} - 1\right|_{\beta} \le N(|H|_0 + |\nabla H|_{\beta\vee 1-1}).$$

The latter inequality was proved in part (3) and the first inequality follows from part (2) and Lemma 3.4.9. □

Remark 3.4.11. Let $H : \mathbf{R}^{d_1} \to \mathbf{R}^{d_1}$ be continuously differentiable and assume that for all x,

$$|\nabla H(x)| \le \eta < 1.$$

Then for all $x \in \mathbf{R}^{d_1}$,

$$|(I_{d_1} + \nabla H(x))^{-1}| \le |I_{d_1} + \sum_{k=1}^{\infty}(-1)^k\nabla H(x)^k| \le \frac{1}{1-\eta}.$$

3.4.4 Stochastic Fubini theorem

Let $m = (m^{\varrho})_{t\le T}$, $\varrho \in \mathbf{N}$, be a sequence of $\mathbf{F}$-adapted locally square integrable continuous martingales issuing from zero such that $\mathbf{P}$-a.s. for all $t \in [0,T]$, $\langle m^{\varrho_1}, m^{\varrho_2}\rangle_t = 0$ for $\varrho_1 \neq \varrho_2$ and $\langle m^{\varrho}\rangle_t = N_t$ for $\varrho \in \mathbf{N}$, where N_t is a $\mathcal{P}_T$-measurable continuous increasing processes issuing from zero. Let $\eta(dt,dz)$ be a $\mathbf{F}$-adapted integer-valued random measure on $([0,T]\times E, \mathcal{B}([0,T])\otimes\mathcal{E})$, where $(U,\mathcal{U})$ is a Blackwell space. We assume that $\eta(dt,dz)$ is optional, $\tilde{\mathcal{P}}_T$-sigma-finite, and quasi-left continuous. Thus, there exists a unique (up to a $\mathbf{P}$-null set) dual predictable projection (or compensator) $\eta^p(dt,dz)$ of $\eta(dt,dz)$ such that $\mu(\omega,\{t\}\times U) = 0$ for all ω and t. We refer the reader to Chapter II, Section 1, in [JS03] for any unexplained concepts relating to random measures.

Let (X,Σ,μ) be a sigma-finite measure space; that is, there is an increasing sequence of Σ-measurable sets X_n, $n \in \mathbf{N}$, such that $X = \cup_{n=1}^{\infty}X_n$ and $\mu(X_n) < \infty$ for each n. Let $f : \Omega\times[0,T]\times X \to \mathbf{R}^{d_2}$ be $\mathcal{R}_T\otimes\Sigma$-measurable, $g : \Omega\times[0,T]\times X \to \ell_2(\mathbf{R}^{d_2})$ be $\mathcal{R}_T\otimes\Sigma/\mathcal{B}(\ell_2(\mathbf{R}^{d_2}))$-measurable, and $h : \Omega\times[0,T]\times X\times U \to \mathbf{R}^{d_2}$ be $\mathcal{P}_T\otimes\Sigma\otimes\mathcal{U}$-measurable. Moreover, assume that for all $t \in [0,T]$ and $x \in X$, $\mathbf{P}$-a.s.

$$\int_{]0,T]}|g_t(x)|^2 dN_t + \int_{]0,T]}\int_U |h_t(x,z)|^2\eta^p(dt,dz) < \infty.$$

Let $F = F_t(x) : \Omega \times [0,T] \times X \to \mathbf{R}^{d_2}$ be $\mathcal{O}_T \otimes \mathcal{B}(X)$-measurable and assume that for $d\mathbf{P}\mu$-almost all $(t,x) \in [0,T] \times X$,

$$F_t(x) = \int_{]0,t]} g_s^{\varrho}(x) dm_s^{\varrho} + \int_{]0,t]} \int_U h_s(x,z) \tilde{\eta}(dt,dz),$$

where $\tilde{\eta}(dt,dz) = \eta(dt,dz) - \eta^p(dt,dz)$.

The following version of the stochastic Fubini theorem is a straightforward extension of Lemma 2.6 [Kry11] and Corollary 1 in [Mik83]. See also Proposition 3.1 in [Zho13], Theorem 2.2 in [Ver12], and Theorem 1.4.8 in [Roz90]. Indeed, to prove it for a bounded measure we can use a monotone class argument as in Theorem 64 in [Pro05]. To handle the general setting with possibly infinite μ, we use assumptions (ii) and (iii) below and take limits on the sets X_n using the Lenglart domination lemma (Theorem 1.4.5 in [LS89]) and the following well-known inequalities:

$$\mathbf{E} \sup_{t \le T} \left| \int_{]0,t]} g_s^{\varrho} dm_s^{\varrho} \right| \le N \mathbf{E} \left(\int_{]0,T]} |g_t(x)|^2 dm_t^{\varrho} \right)^{1/2}$$

$$\mathbf{E} \sup_{t \le T} \left| \int_{]0,t]} \int_U h_t(x,z) \tilde{\eta}(dt,dz) \right| \le N \mathbf{E} \left(\int_{]0,T]} \int_U |h_t(x,z)|^2 \eta^p(dt,dz) \right)^{1/2},$$

where $\tau \le T$ is an arbitrary stopping time and $N = N(T)$ is a constant independent of g and h.

Proposition 3.4.12 (c.f. Corollary 1 in [Mik83] and Lemma 2.6 in [Kry11]). *Assume that*

(1) $\mathbf{P}$*-a.s. for all* $n \ge 1$,

$$\int_{X_n} \left(\int_{]0,T]} |g_t(x)|^2 dN_t \right)^{1/2} \mu(dx) + \int_{X_n} \left(\int_{]0,T]} \int_{U_1} |h_t(x,z)|^2 \eta^p(dt,dz) \right)^{1/2} \mu(dx) < \infty;$$

(2) $\mathbf{P}$*-a.s.*

$$\int_{]0,T]} \left(\int_X |g_t(x)| \mu(dx) \right)^2 dt + \int_{]0,T]} \int_U \left(\int_X |h_t(x,z)| \mu(dx) \right)^2 \eta^p(dt,dz);$$

(3) $\mathbf{P}$*-a.s. for al* $t \in [0,T]$,

$$\int_X |F_t(x)| \mu(dx) < \infty.$$

Then $\mathbf{P}$*-a.s. for all* $t \in [0,T]$,

$$\int_X F_t(x) \mu(dx) = \int_{]0,t]} \int_X g_s^{\varrho}(x) \mu(dx) dm_s^{\varrho} + \int_{]0,t]} \int_U \int_X h_s(x,z) \mu(dx) \tilde{\eta}(dr,dz).$$

We obtain the following corollary by applying Minkowski's integral inequaility.

Corollary 3.4.13. *Assume that* **P***-a.s.*

$$\int_X \left(\int_{]0,T]} |g_t(x)|^2 dN_t \right)^{1/2} \mu(dx) + \int_X \left(\int_{]0,T]} \int_{U_1} |h_t(x,z)|^2 \eta^p(dt,dz) \right)^{1/2} \mu(dx) < \infty. \quad (3.4.6)$$

Then **P***-a.s. for all* $t \in [0,T]$,

$$\int_X F_t(x)\mu(dx) = \int_{]0,t]} \int_X g_s^\varrho(x)\mu(dx) dm_s^\varrho + \int_{]0,t]} \int_U \int_X h_s(x,z)\mu(dx)\tilde{\eta}(dr,dz).$$

Remark 3.4.14. If μ is a finite-measure and **P**-a.s.

$$\int_X \int_{]0,T]} |g_t(x)|^2 dN_t \mu(dx) + \int_X \int_{]0,T]} \int_{U_1} |h_t(x,z)|^2 \eta^p(dt,dz)\mu(dx) < \infty,$$

then (3.4.6) holds by Hölder's inequality.

3.4.5 Itô-Wentzell formula

Definition 3.4.15. We say that an $\mathbf{R}^{d_1}$-valued **F**-adapted quasi-left continuous semimartingale $L_t = (L_t^k)_{1\le k\le d_1}$, $t \ge 0$, is of α-order for $\alpha \in (0,2]$, if **P**-a.s. for all $t \ge 0$,

$$\sum_{s\le t} |\Delta L_s|^\alpha < \infty$$

and

$$\begin{aligned}
L_t &= L_0 + \int_{]0,t]} \int_{\mathbf{R}_0^{d_1}} z p^L(ds,dz), \text{ if } \alpha \in (0,1),\\
L_t &= L_0 + A_t + \int_{]0,t]} \int_{|z|\le 1} z q^L(ds,dz) + \int_{]0,t]} \int_{|z|>1} z p^L(ds,dz), \text{ if } \alpha \in [1,2),\\
L_t &= L_0 + A_t + L_t^c + \int_{]0,t]} \int_{|z|\le 1} z q^L(ds,dz) + \int_{]0,t]} \int_{|z|>1} z p^L(ds,dz), \text{ if } \alpha = 2,
\end{aligned}$$

where $p^L(dt,dz)$ is the jump measure of L with dual predictable projection $\pi^L(dt,dz)$, q^L $(dt,dz) = p^L(dt,dz) - \pi^L(dt,dz)$ is a martingale measure, $A_t = (A_t^i)_{1\le i\le d_1}$ is a continuous process of finite variation with $A_0 = 0$, and $L_t^c = (L_t^{c;i})_{1\le i\le d_1}$ is a continuous local martingale issuing from zero.

Set $(w^\varrho)_{\varrho\in\mathbf{N}} = (w^{1;\varrho})_{\varrho\in\mathbf{N}}$, $(Z, \mathcal{Z}, \pi) = (\mathcal{Z}^1, \mathcal{Z}^1, \pi^1)$, $p(dt,dz) = p^1(dt,dz)$, and $q(dt,dz)$ $= q^1(dt,dz)$. Also, set $D = D^1$, $E = E^1$, and assume $Z = D \cup E$.

Let $f : \Omega \times [0,T] \times \mathbf{R}^{d_1} \to \mathbf{R}^{d_2}$ be $\mathcal{R}_T \otimes \mathcal{B}(\mathbf{R}^{d_1})$-measurable, $g : \Omega \times [0,T] \times \mathbf{R}^{d_1} \to \ell_2(\mathbf{R}^{d_2})$ be $\mathcal{R}_T \otimes \mathcal{B}(\mathbf{R}^{d_1})/\mathcal{B}(\ell_2(\mathbf{R}^{d_2}))$-measurable, and $h : \Omega \times [0,T] \times \mathbf{R}^{d_1} \times Z \to \mathbf{R}^{d_2}$ be

$\mathcal{P}_T \otimes \mathcal{B}(\mathbf{R}^{d_1}) \otimes \mathcal{Z}$-measurable. Moreover, assume that, $\mathbf{P}$-a.s. for all $x \in \mathbf{R}^{d_1}$,

$$\int_{]0,T]} |f_t(x)|dt + \int_{]0,T]} |g_t(x)|^2 dt < \infty$$
$$+ \int_{]0,T]} \int_D |h_t(x,z)|^2 \pi(dz)dt + \int_{]0,T]} \int_E |h_t(x,z)|\pi(dz)dt < \infty.$$

Let $F = F_t(x) : \Omega \times [0,T] \times \mathbf{R}^{d_1} \to \mathbf{R}^{d_2}$ be $\mathcal{O}_T \otimes \mathcal{B}(\mathbf{R}^{d_1})$-measurable and assume that for all x, $\mathbf{P}$-a.s. for all t,

$$F_t(x) = F_0(x) + \int_{]0,t]} f_s(x)ds + \int_{]0,t]} g_s^{\varrho}(x)dw_s^{\varrho}$$
$$+ \int_{]0,t]} \int_Z h_s(x,z)[\mathbf{1}_D(z)q(ds,dz) + \mathbf{1}_E(z)p(ds,dz)].$$

For each $n \in \{1,2\}$, let $C^n_{loc}(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$ be space of n-times continuously differentiable functions $f : \mathbf{R}^{d_1} \to \mathbf{R}^{d_2}$. We now state our version of the Itô-Wentzell formula. For each ω, t and x, we denote $\Delta F(x) = F_t(x) - F_{t-}(x)$.

Proposition 3.4.16 (cf. Proposition 1 in [Mik83]). *Let $(L_t)_{t\geq 0}$ be an $\mathbf{R}^{d_1}$-valued quasi-left continuous semimartingale of order $\alpha \in (0,2]$. Assume that:*

(1) (a) $\mathbf{P}$-a.s. $F \in D([0,T]; C^{\alpha}_{loc}(\mathbf{R}^d;\mathbf{R}^m)$ if α is fractional and $F \in D([0,T]; C^{\alpha}_{loc}(\mathbf{R}^d;\mathbf{R}^m)$ if $\alpha = 1,2$;

(b) for $d\mathbf{P}dt$-almost-all $(\omega,t) \in \Omega \times [0,T]$, $f_t(x)$ and $g_t(x) = (g_t^{i\varrho}(x))_{\varrho\in\mathbf{N}} \in \ell_2(\mathbf{R}^{d_2})$ are continuous in x and

$$d\mathbf{P}dt - \lim_{y\to x}\left[\int_D |h_t(y,z) - h_t(x,z)|^2\pi(dz) + \int_E |h_t(y,z) - h_t(x,z)|\pi(dz)\right] = 0;$$

(c) for all $\varrho \in \mathbf{N}$ and $i \in \{1,\ldots,d_1\}$ and for $d\mathbf{P}d|\langle L^{c;i}, w^{\varrho}\rangle|_t$-almost-all $(\omega,t) \in \Omega \times [0,T]$, $g_t^{i\varrho} \in C^1_{loc}(\mathbf{R}^d;\mathbf{R})$, if $\alpha = 2$, ;

(2) for all compact subsets K of $\mathbf{R}^{d_1}$, $\mathbf{P}$-a.s.

$$\int_{]0,T]} \sup_{x\in K}\left(|f_t(x)| + |g_t(x)|^2 + \int_D |h_t(x,z)|^2\pi(dz) + \int_E |h_t(x,z)|\pi(dz)\right)dt < \infty,$$

$$\sum_{\varrho\in\mathbf{N}} \int_{]0,T]} \sup_{x\in K} |\nabla g_t^{i\varrho}(x)| d|\langle L^{c;i}, w^{\varrho}\rangle|_t + \sum_{t\leq T} |\Delta F_t|_{\alpha\wedge 1;K} |\Delta L_t|^{\alpha\wedge 1} < \infty.$$

Then $\mathbf{P}$-a.s for all $t \in [0,T]$,

$$F_t(L_t) = F_0(L_0) + \int_{]0,t]} f_s(L_s)ds + \int_{]0,t]} g_s^{\varrho}(L_s)dw_s^{\varrho}$$

$$
\begin{aligned}
&+\int_{]0,t]}\int_Z h_s(L_{s-},z)[\mathbf{1}_D(z)q(dr,dz)+\mathbf{1}_E(z)p(dr,dz)]\\
&+\int_{]0,t]}\partial_i F_{s-}(L_{s-})[\mathbf{1}_{[1,2]}(\alpha)dA^i_s+\mathbf{1}_{\{2\}}(\alpha)dL^{c;i}_s]\\
&+\sum_{s\le t}(F_{s-}(L_s)-F_{s-}(L_{s-})-\mathbf{1}_{[1,2]}(\alpha)\nabla F_{s-}(L_{s-})\Delta L_s)\\
&+\mathbf{1}_{\{2\}}(\alpha)\frac12\int_{]0,t]}\partial_{ij}F_s(L_s)d\langle L^{c;i},L^{c;j}\rangle_s \qquad (3.4.7)\\
&+\mathbf{1}_{\{2\}}(\alpha)\int_{]0,t]}\partial_i g^{\varrho}_s(L_s)d\langle w^{\varrho},L^{c;i}\rangle_s+\sum_{s\le t}(\Delta F_s(L_s)-\Delta F_s(L_{s-}))\,.
\end{aligned}
$$

Proof. Since both sides have identical jumps and we can always interlace a finite set of jumps, we may assume that $|\Delta L_t|\le 1$ for all $t\in[0,T]$; that is, it is enough to prove the statement for $\tilde L_t=L_t-\sum_{s\le t}\mathbf{1}_{[1,\infty)}(|\Delta L_s|)\Delta L_s$, $t\in[0,T]$. It suffices to assume that for some K and all ω, $|L_0|\le K$. For each $R>K$, let

$$
\tau_R=\inf\left(t\in[0,T]:|A|_t+|\langle L^c\rangle|_t+\sum_{s\le t}|\Delta L_s|^{\alpha}+|L_t|>R\right)\wedge T
$$

and note that $\mathbf{P}$-a.s. $\tau_R\uparrow T$ as R tends to infinity. If instead of L,f,g,h, and F, we take $L_{\cdot\wedge\tau_R}$, $f\mathbf{1}_{(0,\tau_R]}$, $g^{\varrho}\mathbf{1}_{(0,\tau_R]}$, $h\mathbf{1}_{(0,\tau_R]}$, $F\mathbf{1}_{(0,\tau_R]}$, then the assumptions of the proposition hold for this new set of processes. Moreover, if we can prove (3.4.7) for this new set of processes, then by taking the limit as R tends to infinity, we obtain (3.4.7). Therefore, we may assume that for some $R>0$, $\mathbf{P}$-a.s. for all $t\in[0,T]$,

$$
|A|_t+|\langle L^c\rangle|_t+\sum_{s\le t}|\Delta L_s|^{\alpha}+|L_t|\le R. \qquad (3.4.8)
$$

Let $\phi\in C^\infty_c(\mathbf{R}^{d_1},\mathbf{R})$ have support in the unit ball in $\mathbf{R}^{d_1}$ and satisfy $\int_{\mathbf{R}^{d_1}}\phi(x)dx=1$, $\phi(x)=\phi(-x)$, and $\phi(x)\ge 0$, for all $x\in\mathbf{R}^{d_1}$. For each $\varepsilon\in(0,1)$, let $\phi_\varepsilon(x)=\varepsilon^{-d}\phi\left(x/\varepsilon\right),x\in\mathbf{R}^{d_1}$. By Itô's formula, for all $x\in\mathbf{R}^{d_1}$, $\mathbf{P}$-a.s. for all $t\in[0,T]$,

$$
\begin{aligned}
F_t(x)\phi_\varepsilon(x-L_t)=F_0(x)\phi_\varepsilon(x-L_0)&-\int_{]0,t]}F_{s-}(x)\partial_i\phi_\varepsilon(x-L_{s-})dL^i_s\\
&+\int_{]0,t]}\phi_\varepsilon\left(x-L_s\right)f_s(x)ds+\int_{]0,t]}\phi_\varepsilon\left(x-L_s\right)g^{\varrho}_s(x)dw^{\varrho}_s\\
&+\mathbf{1}_{\{2\}}(\alpha)\frac12\int_{]0,t]}F_s(x)\partial_{ij}\phi_\varepsilon(x-L_s)d\langle L^{c;i},L^{c;j}\rangle_s\\
&+\mathbf{1}_{\{2\}}(\alpha)\int_{]0,t]}g^{\varrho}_s(x)\partial_i\phi_\varepsilon(x-L_s)d\langle w^{\varrho},L^{c;i}\rangle_s\\
&+\int_{]0,t]}\int_Z\phi_\varepsilon\left(x-L_{s-}\right)h_s(x,z)[\mathbf{1}_D(z)q(dr,dz)+\mathbf{1}_E(z)p(dr,dz)]
\end{aligned}
$$

$$+\sum_{s\le t}\Delta F_s(x)\,(\phi_\varepsilon(x-L_s)-\phi_\varepsilon\,(x-L_{s-}))$$
$$+\sum_{s\le t}F_{s-}(x)\,(\phi_\varepsilon(x-L_s)-\phi_\varepsilon(x-L_{s-})+\partial_i\phi_\varepsilon\,(x-L_{s-})\,\Delta L_s)\,.$$

Appealing to assumption (2) and (3.4.8) (i.e. for the integrals against F), we integrate both sides of the above in x, apply Corollary 3.4.13 (see, also, Remark 3.4.14) and the deterministic Fubini theorem, and then integrate by parts to get that $\mathbf{P}$-a.s. for all $t\in[0,T]$,

$$\begin{aligned}F_t^{(\varepsilon)}(L_t) &= F_0^{(\varepsilon)}(L_0)+\int_{]0,t]}\nabla F_{s-}^{(\varepsilon)}(L_{s-})[\mathbf{1}_{[1,2]}(\alpha)dA_s^i+\mathbf{1}_{\{2\}}(\alpha)dL_s^{c;i}]+\int_{]0,t]}f_s^{(\varepsilon)}(L_s)dr\\
&+\int_{]0,t]}g_s^{(\varepsilon)}(L_s)dw_s^{\varrho}+\int_{]0,t]}\int_Z h_s^{(\varepsilon)}(L_{s-},z)[\mathbf{1}_D(z)q(dr,dz)+\mathbf{1}_E(z)p(dr,dz)]\\
&+\mathbf{1}_{\{2\}}(\alpha)\frac{1}{2}\int_{]0,t]}\partial_{ij}F_s^{(\varepsilon)}(L_s)d\langle L^{c;i},L^{c;j}\rangle_s+\mathbf{1}_{\{2\}}(\alpha)\int_{]0,t]}\partial_i g_s^{(\varepsilon);\varrho}(L_s)d\langle w^{\varrho},L^{c;i}\rangle_s\\
&+\sum_{s\le t}\left(\Delta F_s^{(\varepsilon)}(L_s)-\Delta F_s^{(\varepsilon)}(L_{s-})\right)\\
&+\sum_{s\le t}\left(F_{s-}^{(\varepsilon)}(L_s)-F_{s-}^{(\varepsilon)}(L_{s-})-\mathbf{1}_{[1,2]}(\alpha)\nabla F_{s-}^{(\varepsilon)}(L_{s-})\Delta L_s\right)\end{aligned}\tag{3.4.9}$$

where for all ω,t,x, and z,

$$F_t^{(\varepsilon)}(x):=\phi_\varepsilon*F_t(x),\quad f_t^{(\varepsilon)}=\phi_\varepsilon*f_t(x),\quad g_t^{(\varepsilon);\varrho}(x)=\phi_\varepsilon*g_t^{\varrho}(x),\quad h_t^{(\varepsilon)}(x,z)=\phi_\varepsilon*h_t(x,z),$$

and $*$ denotes the convolution operator on $\mathbf{R}^{d_1}$. Let $B_{R+1}=\{x\in\mathbf{R}^{d_1}:|x|\le R+1\}$. Owing to assumption (1)(a) and standard properties of mollifiers, for any multi-index γ with $|\gamma|\le\alpha$, $\mathbf{P}$-a.s. for all t,

$$|\partial^\gamma F_t^{(\varepsilon)}(L_t)|\le\sup_{t\le T}\sup_{x\in B_{R+1}}|\partial^\gamma F_t(x)|<\infty$$

and for all x,

$$d\mathbf{P}dt-\lim_{\varepsilon\downarrow 0}|\partial^\gamma F_t^{(\varepsilon)}(x)-\partial^\gamma F_t^{(\varepsilon)}(x)|=0.$$

Similarly, by assumption 1(b), $d\mathbf{P}dt$-almost-all $(\omega,t)\in\Omega\times[0,T]$,

$$|f_t^{(\varepsilon)}(L_t)|\le\sup_{x\in B_{R+1}}|f_t(x)|<\infty,\quad |g_t^{(\varepsilon)}(L_t)|\le\sup_{x\in B_{R+1}}|g_t(x)|<\infty,$$
$$\int_D|h_t^{\varepsilon}(L_t,z)|^2\pi(dz)\le\sup_{x\in B_{R+1}}\int_D|h_t(x,z)|^2\pi(dz),$$
$$\int_E|h_t^{\varepsilon}(L_t,z)|\pi(dz)\le\sup_{x\in B_{R+1}}\int_E|h_t(x,z)|\pi(dz)$$

and for all x,

$$d\mathbf{P}dt - \lim_{\varepsilon\downarrow 0} |f_t^{(\varepsilon)}(x) - f_t(x)| = 0, \quad d\mathbf{P}dt - \lim_{\varepsilon\to 0} |g_t^{(\varepsilon)}(x) - g_t(x)| = 0$$

and

$$d\mathbf{P}dt - \lim_{\varepsilon\downarrow 0} \int_Z [\mathbf{1}_D(z)|h_t^{(\varepsilon)}(x,z) - h_t(x,z)|^2 + \mathbf{1}_E(z)|h_t^{(\varepsilon)}(x,z) - h_t(x,z)|]\pi(dz) = 0,$$

where in the last-line we have also used Minkowski's integral inequality and a standard mollifying convergence argument. Using assumption 1(d), for all $\varrho \in \mathbf{N}$ and $i \in \{1,\dots,d_1\}$ and for $d\mathbf{P}d|\langle L^{c;i}, w^\varrho\rangle|_t$-almost-all $(\omega,t) \in \Omega \times [0,T]$

$$|\nabla g_t^{(\varepsilon);i\varrho}(L_t)| \le \sup_{x\in B_{R+1}} |\nabla g_t^{i\varrho}(x)|$$

and for all x,

$$d\mathbf{P}d|\langle L^{c;i}, w^\varrho\rangle|_t - \lim_{\varepsilon\to 0} |\nabla g_t^{(\varepsilon);i\varrho}(x) - \nabla g_t^{i\varrho}(x)| = 0, \quad \text{if } \alpha = 2.$$

Owing to assumption 1(a) and (3.4.8), $\mathbf{P}$-a.s.

$$\begin{aligned}&\sum_{s\le t} |F_{s-}^{(\varepsilon)}(L_s) - F_{s-}^{(\varepsilon)}(L_{s-}) - \mathbf{1}_{[1,2]}(\alpha)\nabla F_{s-}^{(\varepsilon)}(L_{s-})\Delta L_s| \\ &\quad \le \sup_{t\le T} |F_t|_{\alpha;B_{R+1}} \sum_{s\le t} |\Delta L_s|^\alpha \le R \sup_{t\le T} |F_t|_{\alpha;B_{R+1}}.\end{aligned}$$

Since $\mathbf{P}$-a.s. $F \in D([0,T]; C^\alpha(\mathbf{R}^d;\mathbf{R}^m))$, it follows that for all x, $\mathbf{P}$-a.s. for all t,

$$\lim_{\varepsilon\downarrow 0} |\Delta F_t^\varepsilon(x) - \Delta F_t(x)| = 0.$$

By assumption (2), $\mathbf{P}$-a.s for all t, we have

$$\sum_{s\le t} \left(\Delta F_s^{(\varepsilon)}(L_{s-} + \Delta L_s) - \Delta F_s^{(\varepsilon)}(L_{s-})\right) \le \sum_{s\le t} |\Delta F_t|_{\alpha\wedge 1;B_{R+1}} |\Delta L_s|^{\alpha\wedge 1}.$$

Combining the above and using assumptions (1)(a) and (2) and the bounds given in (3.4.8) and the deterministic and stochastic dominated convergence theorem, we obtain convergence of all the terms in (3.4.9), which complete the proof. □

Chapter 4
The L^2-Sobolev theory for parabolic SIDEs

4.1 Introduction

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space with the filtration $\mathbf{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ of sigma-algebras satisfying usual conditions. In a triple of Hilbert spaces $(H^{\alpha+\mu}, H^{\alpha}, H^{\alpha-\mu})$ with parameters $\mu \in (0,1]$ and $\alpha \geq \mu$, we consider a linear stochastic evolution equation given by

$$du_t = (\mathcal{L}_t u_t + f_t)\, dV_t + (\mathcal{M}_t u_{t-} + g_t)\, dM_t, \quad t \leq T, \tag{4.1.1}$$
$$u_0 = \varphi,$$

where V_t is a continuous non-decreasing process, M_t is a cylindrical square integrable martingale, $\mathcal{L}$ and $\mathcal{M}$ are linear adapted operators, and ϕ, f, and g are adapted input functions.

By virtue of Theorems 2.9 and 2.10 in [Gyö82], under some suitable conditions on the data φ, f and g, if $\mathcal{L}$ satisfies a growth assumption and $\mathcal{L}$ and $\mathcal{M}$ satisfy a coercivity condition in the triple $(H^{\alpha+\mu}, H^{\alpha}, H^{\alpha-\mu})$, then there exists a unique solution $(u_t)_{t\leq T}$ of (4.1.1) that is strongly càdlàg in H^{α} and belongs to $L^2(\Omega\times[0,T], \mathcal{O}_T, dV_t d\mathbf{P}; H^{\alpha+\mu})$, where $\mathcal{O}_T$ is the optional sigma-algebra on $\Omega\times[0,T]$. In this chapter, under a weaker assumption than coercivity (see Assumption 4.2.1(α,μ)) and using the method of vanishing viscosity, we prove that there exists a unique solution $(u_t)_{t\leq T}$ of (4.1.1) that is strongly càdlàg in $H^{\alpha'}$ for all $\alpha' < \alpha$ and belongs to $L^2(\Omega\times[0,T], dV_t d\mathbf{P}; H^{\alpha})$. Furthermore, under some additional assumptions on the operators $\mathcal{L}$ and $\mathcal{M}$ we can show that the solution u is weakly càdlàg in H^{α}.

The variational theory of deterministic degenerate linear elliptic and parabolic PDEs was established by O.A. Oleinik and E.V. Radkevich in [Ole65] and [OR71]. In [Par75], É. Pardoux developed the variational theory of monotone stochastic evolution equations, which was extended in [KR77, KR79, GK81, Gyö82] by N.V. Krylov, B.L Rozovskiĭ, and I. Gyöngy. Degenerate parabolic stochastic partial differential equations (SPDEs) driven by continuous noise were first investigated by N.V. Krylov and B.L. Rozovskiĭ in [KR82]. These types of equations arise in the theory of non-linear filtering of continuous diffusion processes as the Zakai equation and as equations governing the inverse flow of continuous diffusions. In [GGK14], the solvability of systems of linear SPDEs in Sobolev spaces was proved by M. Gerencsér, I. Gyöngy, and N.V. Krylov, and a small gap in the proof of the main result of [KR82] was fixed. In Chapters 2, 3, and 4 of [Roz90], B.L. Rozovskiĭ offers a unified presentation and extension of earlier results on the variational framework of linear stochastic evolution systems and SPDEs driven by continuous martingales (e.g. [Par75, KR77, KR79, KR82]). Our existence and uniqueness result on degenerate linear stochastic evolution equations driven by jump processes (Theorem 4.3.2) extends Theorem 2 in Chapter-3-Section 2.2 of [Roz90] to include the important case of equations driven by jump processes. It is also worth mentioning that the semigroup approach for non-degenerate SPDEs driven by Lévy processes is well-studied (see, e.g. [PZ07, PZ13]).

As a special case of (4.1.1), we will consider a system of SIDEs. Before introducing the equation, let us describe our driving processes. Let $\eta(dt,dz)$ be an integer-valued random measure on $(\mathbf{R}_+\times Z,\mathcal{B}(\mathbf{R}_+)\otimes\mathcal{Z})$ with predictable compensator $\pi_t(dz)dV_t$. Let $\tilde{\eta}(dt,dz)=\eta(dt,dz)-\pi_t(dz)dt$ be the martingale measure corresponding to $\eta(dt,dz)$. Let $(Z^2,\mathcal{Z}^2)$ be a measurable space with $\mathcal{R}_T$-measurable family $\pi^2_t(dz)$ of sigma-finite random measures on Z. Let w^ϱ_t, $t\geq 0$, $\varrho\in\mathbf{N}$, be a sequence of continuous local uncorrelated martingales such that $d\langle w^\varrho\rangle_t=dV_t$, for all $\rho\in\mathbf{N}$. Let $d_1,d_2\in\mathbf{N}$. For convenience, we set $(Z^1,\mathcal{Z}^1)=(Z,\mathcal{Z})$ and $\pi^1_t=\pi_t$. We consider the d_2-dimensional system of SIDEs on $[0,T]\times\mathbf{R}^{d_1}$ given by

$$
\begin{aligned}
du^l_t &= \Big((\mathcal{L}^{1;l}_t+\mathcal{L}^{2;l}_t)u_t+b^i_t\partial_i u^l_t+c^{l\bar{l}}_t u^{\bar{l}}_t(x)+f^l_t\Big)dV_t+(\mathcal{N}^{l\varrho}_t u_t+g^{l\varrho}_t)dw^\varrho_t \\
&\quad+\int_{Z^1}\Big(\mathcal{I}^l_{t,z}u^{\bar{l}}_{t-}+h^l_t(z)\Big)\tilde{\eta}(dt,dz), \\
u^l_0 &= \varphi^l,\quad l\in\{1,\dots,d_2\},
\end{aligned}
\tag{4.1.2}
$$

where for $k\in\{1,2\}$, $l\in\{1,\dots,d_2\}$, and $\phi\in C^\infty_c(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$,

$$
\begin{aligned}
\mathcal{L}^{k;l}_t\phi(x) &:= \frac{1}{2}\sigma^{k;i\varrho}_t(x)\sigma^{k;j\varrho}_t(x)\partial_{ij}\phi^l(x)+\sigma^{k;i\varrho}_t(x)\upsilon^{k;l\bar{l}\varrho}_t(x)\partial_i\phi^{\bar{l}}(x) \\
&\qquad\int_{Z^k}\Big(\big(\delta_{l\bar{l}}+\rho^{k;l\bar{l}}_t(x,z)\big)\Big(\phi^{\bar{l}}(x+\zeta^k(x,z))-\phi^{\bar{l}}(x)\Big)-\zeta^{k;i}_t(x,z)\partial_i\phi^l(x)\Big)\pi^k_t(dz) \\
\mathcal{N}^{l\varrho}_t\phi(x) &:= \sigma^{1;i\varrho}_t(x)\partial_i\phi^l(x)+\upsilon^{1;l\bar{l}\varrho}_t(x)\phi^{\bar{l}}(x),\ \varrho\in\mathbf{N}, \\
\mathcal{I}^l_{t,z}\phi(x) &:= (\delta_{l\bar{l}}+\rho^{1;l\bar{l}}_t(x,z))\phi^{\bar{l}}(x+\zeta^1_t(x,z))-\phi^l(x),
\end{aligned}
$$

and where $\delta_{l\bar{l}}$ is the Kronecker delta (i.e. $\delta_{l\bar{l}}=1$ if $l=\bar{l}$ and $\delta_{l\bar{l}}=0$ otherwise). The summation convention with respect to repeated indices is used here and below; summation over i is performed over the set $\{1,\dots,d_1\}$ and the summation over $l,\bar{l}$ is performed over the set $\{1,\dots,d_2\}$. Without the noise term $\tilde{\eta}(dt,dz)$ and integro-differential operators in $\mathcal{L}^1$ and $\mathcal{L}^2$, equation (4.1.2) has been well-studied (see, e.g. [KR82], [Roz90] (Chapter 3), and the recent paper [GGK14]). The choice to use the notation $\zeta^k(x,z)$ rather than the traditional $H^k(x,z)$ (as is used above) in this chapter due to the conflict of notation with our spaces.

Let $(H^\alpha(\mathbf{R}^{d_1};\mathbf{R}^{d_2}))_{\alpha\in\mathbf{R}}$ be the L^2-Sobolev-scale. For each $m\in\mathbf{N}$, using our theorem on degenerate stochastic evolution equations discussed above, under suitable measurability and regularity conditions on the coefficients, initial condition, and free terms, we derive the existence of a unique solution $(u_t)_{t\le T}$ of (4.1.2) that is weakly càdlàg in $H^m(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$, strongly càdlàg in $H^\alpha(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$ for all $\alpha<m$, and belongs to $L^2(\Omega\times[0,T],\mathcal{O}_T,dV_td\mathbf{P};H^m(\mathbf{R}^{d_1};\mathbf{R}^{d_2}))$.

Degenerate SIDEs of type (4.1.2) arise in the theory of non-linear filtering of semimartingales as the Zakai equation and as the equations governing the inverse flow of jump diffusion processes. We constructed solutions of the above equation (with $\pi_t(dz)$ deterministic and independent of time) using the method

This chapter is organized as follows. We derive our existence and uniqueness result for (4.1.1) in Section 4.2 and for (4.1.2) in Section 4.3.

4.2 Degenerate linear stochastic evolution equations

4.2.1 Basic notation and definitions

All vector spaces considered in this paper are assumed to have base field $\mathbf{R}$. We also assume that all Hilbert spaces are separable. For a Hilbert space H, we denote by H^* the dual of H and by $\mathcal{B}(H)$ the Borel sigma-algebra of H. Unless otherwise stated, the norm and inner product of a Hilbert space H are denoted by $|\cdot|_H$ and $(\cdot,\cdot)_H$, respectively. For Hilbert spaces H and U and a bounded linear map $L:H\to U$, we denote by L^* the Hilbert adjoint of L. Whenever we say that a map F from a sigma-finite measure space $(S,\mathcal{S},\mu)$ to a Hilbert space H is $\mathcal{S}$-measurable without specifying the sigma-algebra on H, we always mean that F is $\mathcal{S}/\mathcal{B}(H)$-measurable. For any Hilbert space H and sigma-finite measure space $(S,\mathcal{S},\nu)$, we denote by $L^2(S,\mathcal{S},\mu;H)$ the linear space of all $\mathcal{S}$-measurable functions $F:S\to H$ such that

$$|F|_{L^2(S,\mathcal{S},\nu;H)}=\int_S|F(s)|_H^2\nu(ds)<\infty,$$

where we identify functions $F,G:S\to H$ that are equal μ-almost-everywhere (ν-a.e.). The linear space $L^2(S,\mathcal{S},\nu;H)$ is a Hilbert space when endowed with the inner product

$$(F,G)_{L^2(S,\mathcal{S},\nu;H)}:=\int_S(F(s),G(s))_H\nu(ds).$$

We use the notation $N=N(\cdot,\cdots,\cdot)$ below to denote a positive constant depending only on the quantities appearing in the parentheses. In a given context, the same letter is often used to denote different constants depending on the same parameter. All the stochastic processes considered below are (at least) $\mathbf{F}$-adapted unless explicitly stated otherwise. Furthermore, we will often drop the dependence on $\omega \in \Omega$ for random quantities.

In this section, we consider a scale of Hilbert spaces $(H^\alpha)_{\alpha\in\mathbf{R}}$ and a family of operators $(\Lambda^\alpha)_{\alpha\in\mathbf{R}}$ satisfying the following properties:

- for all $\alpha,\beta \in \mathbf{R}$ with $\beta > \alpha$, H^β is densely embedded in H^α;
- for all $\alpha,\beta,\mu \in \mathbf{R}$ with $\alpha < \beta < \mu$ and all $\varepsilon > 0$, there is a constant $N = N(\alpha,\beta,\mu,\varepsilon)$ such that
$$|v|_\beta \le \varepsilon|v|_\mu + N|v|_\alpha, \quad \forall v \in H^\mu; \tag{4.2.1}$$
- $\Lambda^0 = I$; for all $\alpha,\mu \in \mathbf{R}$, $\Lambda^\alpha : H^\mu \to H^{\mu-\alpha}$ is an isomorphism; for all $\alpha,\beta \in \mathbf{R}$, $\Lambda^{\alpha+\beta} = \Lambda^\alpha\Lambda^\beta$;
- for all $\alpha \in \mathbf{R}$, the inner product in H^α is given by $(\cdot,\cdot)_\alpha = (\Lambda^\alpha\cdot,\Lambda^\alpha\cdot)_0$;
- for all $\alpha > 0$, the dual $(H^\alpha)^*$ can be identified with $H^{-\alpha}$ through the duality product given by
$$\langle u, v\rangle_\alpha = \langle u, v\rangle_{H^\alpha,H^{-\alpha}} = \left(\Lambda^\alpha u, \Lambda^{-\alpha}v\right)_0, \quad u \in H^\alpha,\ v \in H^{-\alpha};$$
- We assume that for every $\alpha \ge 0$, Λ^α is selfadjoint as an unbounded operator in H^0 with domain $H^\alpha \subseteq H^0$: i.e. $(\Lambda^\alpha u, v)_0 = (u, \Lambda^\alpha v)_0$ for all $u, v \in H^\alpha$.

Remark 4.2.1. It follows from the above properties that for all $\alpha \in \mathbf{R}$, the H^α norm is given by $|v|_\alpha = |\Lambda^\alpha v|_0$, Λ^α is defined and linear on $\cup_{\beta\in\mathbf{R}}H^\beta$, $\Lambda^{-\alpha} = (\Lambda^\alpha)^{-1}$, and $\Lambda^\alpha\Lambda^\beta = \Lambda^\beta\Lambda^\alpha$, for all $\beta \in \mathbf{R}$. Moreover, for each $\alpha \ge 0$, if $u \in H^\alpha$ and $v \in H^0$, then $\langle u, v\rangle_\alpha = (u, v)_0$.

We will now describe our driving cylindrical martingale $(M_t)_{t\ge 0}$ in (4.1.1)) and the associated stochastic integral. For a more thorough exposition, we refer to [MR99]. Let E be a locally convex quasi-complete topological vector space; all bounded closed subsets of E are complete. Let E^* be its topological dual. Denote by $\langle\cdot,\cdot\rangle_{E^*,E}$ the canonical bilinear form (duality product) on $E^* \times E$. Assume that E^* is weakly separable. Denote by $\mathcal{L}^+(E)$ the space of symmetric non-negative definite forms Q from E^* to E; that is, for all $Q \in \mathcal{L}^+(E)$, we have

$$\langle x, Qy\rangle_{E^*,E} = \langle y, Qx\rangle_{E^*,E}, \text{ and } \langle x, Qx\rangle_{E^*,E} \ge 0, \quad \forall x, y \in E^*.$$

Recall that $\mathcal{P}_T$ is the predictable sigma-algebra on $\Omega \times [0, T]$. We say that a process $Q : \Omega\times[0, T] \to \mathcal{L}^+(E)$ is $\mathcal{P}_T$-measurable if $\langle y, Q_t x\rangle_{E^*,E}$ is $\mathcal{P}_T$-measurable for all $x, y \in E^*$.

Assume that we are given a family of real-valued locally square integrable martingales $M = (M_t(y))_{y\in E^*})_{t\geq 0}$ indexed by E^* and an increasing $\mathcal{P}_T$-measurable process $Q : \Omega \times [0, T] \to \mathcal{L}^+(E)$ such that for all $x, y \in E^*$,

$$M_t(x)M_t(y) - \int_0^t \langle x, Q_s y\rangle_{E^*,E} dV_s, \quad t \geq 0,$$

is a local martingale.

For each $(\omega, t) \in \Omega \times [0, T]$, let $\mathcal{H}_t = \mathcal{H}_{\omega,t}$ be the Hilbert subspace of E defined as the completion of $Q_{\omega,t}E^*$ with respect to the inner product

$$(Q_{\omega,t}x, Q_{\omega,t}y)_{\mathcal{H}_{\omega,t}} := \langle x, Q_{\omega,t}y\rangle, \quad x, y \in E^*.$$

It can be shown that for all $(\omega, t) \in \Omega \times [0, T]$, E^* is densely embedded into $\mathcal{H}_t^*$, the map $Q_t : E^* \to E$ can be extended to the Riesz isometry $Q_t : \mathcal{H}_t^* \to \mathcal{H}_t$ (still denoted Q_t), and the bilinear form $\langle x, Q_t y\rangle_{E^*,E}$, $x, y \in E^*$, can be extended to $\langle x, Q_t y\rangle_{\mathcal{H}_t^*,\mathcal{H}_t}$, $x, y \in \mathcal{H}_t^*$. Note that for all $x, y \in \mathcal{H}_t^*$, we have $(x, y)_{\mathcal{H}_t^*} = \langle x, Q_t y\rangle_{\mathcal{H}_t^*,\mathcal{H}_t}$.

Let $\hat{L}^2_{loc}(Q)$ the space of all processes f such that $f_t \in \mathcal{H}_t^*$, $dV_t d\mathbf{P}$-a.e., $\langle f_t, Q_t y\rangle_{\mathcal{H}_t^*,\mathcal{H}_t}$ is $\mathcal{P}_T$-measurable for all $y \in E^*$, and $\mathbf{P}$-a.s.

$$\int_0^T |f_t|^2_{\mathcal{H}_t^*} dV_t = \int_0^T \langle f_t, Q_t f_t\rangle_{\mathcal{H}_t^*,\mathcal{H}_t} dV_t < \infty.$$

In [MR99], the stochastic integral of $f \in \hat{L}^2_{loc}(Q)$ against M, denoted $\Im_t(f) = \int_0^t f_s dM_s$, $t \geq 0$, was constructed and has the following properties: $(\Im_t(f))_{t\geq 0}$ is a locally square integrable martingale and $\mathbf{P}$-a.s. for all $t \in [0, T]$:

- for all $y \in E^*$, $\Im_t(y) = \int_0^t y dM_s = M_t(y)$ (recall that E^* is embedded into all $\mathcal{H}_s^*$);
- for all $g \in \hat{L}^2_{loc}(Q)$.

$$\langle \Im(f), \Im(g)\rangle_t = \int_0^t (f_s, g_s)_{\mathcal{H}_s^*} dV_s = \int_0^t \langle f_s, Q_s g_s\rangle_{\mathcal{H}_s^*,\mathcal{H}_s} dV_s;$$

- for all bounded $\mathcal{P}_T$-measurable processes $\phi : \Omega \times [0, T] \to \mathbf{R}$,

$$\int_0^t \phi_s d\Im_s(f) = \Im_t(\phi f) = \int_0^t \phi_s f_s dM_s.$$

For a Hilbert space H and $(\omega, t) \in \Omega \times [0, T]$, denote by $L_2(H, \mathcal{H}_t^*)$ the space of all

Hilbert-Schmidt operators $\Psi : H \to \mathcal{H}_t^*$ with norm and inner product given by

$$|\Psi|^2_{L_2(H,\mathcal{H}_t^*)} := \sum_{n=1}^{\infty} |\Psi h^n|^2_{\mathcal{H}_t^*}, \quad (\Psi, \tilde{\Psi})_{L_2(H,\mathcal{H}_t^*)} = \sum_{n=1}^{\infty} (\Psi h^n, \tilde{\Psi} h^n)_{\mathcal{H}_t^*}, \quad \tilde{\Psi} \in L_2(H, \mathcal{H}_t^*),$$

where $(h^n)_{n\in\mathbf{N}}$ is a complete orthogonal system in H. Denote by $L^2_{loc}(H, Q)$ the space of all processes Ψ such that $\Psi_t \in L_2(H, \mathcal{H}_t^*)$, $dV_t d\mathbf{P}$-a.e., $\Psi_t h \in \hat{L}^2(Q)$, for each $h \in H$,, and $\mathbf{P}$-a.s.

$$\int_0^T |\Psi_t|^2_{L_2(H,\mathcal{H}_t^*)} dV_t < \infty.$$

For each $\Psi \in L^2_{loc}(H, Q)$, we define the stochastic integral $\Im_t(\Psi) = \int_0^t \Psi_s dM_s$ as the unique H-valued càdlàg locally square integrable martingale such that $\mathbf{P}$-a.s. for all $t \in [0, T]$ and $h \in H$,

$$(\Im_t(\Psi), h)_H = \int_0^t \Psi_s h dM_s.$$

For all $\Psi, \tilde{\Psi} \in L^2_{loc}(H, Q)$, we have that

$$|\Im_\cdot(\Psi)|^2_H - \int_0^\cdot |\Psi_s|^2_{L_2(H,\mathcal{H}_s^*)} dV_s \;\; and \;\; (\Im_\cdot(\Psi), \Im_\cdot(\tilde{\Psi}))_H - \int_0^\cdot (\Psi_s, \tilde{\Psi}_s)_{L_2(H,\mathcal{H}_s^*)} dV_s$$

are real-valued local martingales. Moreover, for all bounded $\mathcal{P}_T$-measurable H-valued processes $u : \Omega \times [0, T] \to H$, $\mathbf{P}$-a.s. for all $t \in [0, T]$,

$$\int_0^t u_s d\Im_s(\Psi) = \int_0^t \{u_s \Psi_s\}_H dM_s,$$

where for a complete orthogonal system $(\tilde{e}^n_s)_{n\in\mathbf{N}}$ in $\mathcal{H}_s^*$,

$$\{u_s \Psi_s\}_H := \sum_{n=1}^{\infty} (\Psi_s u_s, \tilde{e}^n_s)_{\mathcal{H}_s^*} \tilde{e}^n_s.$$

If H and Y are Hilbert spaces and $L : H \to Y$ is a bounded linear operator and $\Psi \in L^2_{loc}(H, Q)$, then it follows that $L\Im_t(\Psi) = \Im_t(\Psi L^*)$; indeed, for all $y \in Y$, we have

$$(L\Im_t(\Psi), y)_Y = (\Im_t(\Psi), L^* y)_H = \int_0^t \Psi_s L^* y dM_s$$

and $\Psi L^* \in L^2_{loc}(Y, Q)$.

4.2.2 Statement of main results

In this section, for $\mu \in (0,1]$, we consider the linear stochastic evolution equation in the triple $(H^{-\mu}, H^0, H^\mu)$ given by

$$\begin{aligned} du_t &= (\mathcal{L}_t u_t + f_t)\,dV_t + (\mathcal{M}_t u_{t-} + g_t)\,dM_t, \quad t \le T, \\ u_0 &= \varphi, \end{aligned} \tag{4.2.2}$$

where φ is an $\mathcal{F}_0$-measurable H^0-valued random variable and V_t is a continuous non-decreasing process such that $V_t \le C$ for all $(\omega,t) \in \Omega \times [0,T]$, for some positive constant C. Let $\alpha \ge \mu$ be given. We assume that:

(1) the mapping $\mathcal{L} : \Omega \times [0,T] \times H^\mu \to H^{-\mu}$ is linear in H^μ, and for all $v \in H^\mu$, $\mathcal{L}v$ is $\mathcal{R}_T/\mathcal{B}(H^{-\mu})$-measurable; in addition, $dV_t d\mathbf{P}$-a.e., $\mathcal{L}_t v \in H^{\alpha-\mu}$ for all $v \in H^{\alpha+\mu}$;
(2) for $dV_t d\mathbf{P}$-almost-all $(\omega,t) \in \Omega \times [0,T]$, $\mathcal{M}_{\omega,t} : H^\mu \to L_2(H^0, \mathcal{H}^*_{\omega,t})$ is linear, and for all $v \in H^\mu$, $\phi \in H^0$, $y' \in E^*$, $\langle (\mathcal{M}v)\phi, Q_t y'\rangle_{\mathcal{H}^*_t,\mathcal{H}_t}$ is $\mathcal{P}_T$-measurable for all $y' \in E^*$; in addition, $dV_t d\mathbf{P}$-a.e, $\mathcal{M}_t v \in L_2(H^\alpha, \mathcal{H}^*_t)$ for all $v \in H^{\alpha+\mu}$;
(3) the process $f : \Omega \times [0,T] \to H^{\alpha-\mu}$ is $\mathcal{R}_T/\mathcal{B}(H^{\alpha-\mu})$-measurable and $g \in L^2_{loc}(H^\alpha, Q) \cap L^2_{loc}(H^0, Q)$,

Let us introduce the following assumption for $\lambda \in \{0,\alpha\}$. Recall that $(u,v)_\lambda = \left(\Lambda^\lambda u, \Lambda^\lambda v\right)_0$.

Assumption 4.2.1 (λ,μ)**.** *There are positive constants L and K and an $\mathcal{R}_T$-measurable function $\bar f : \Omega \times [0,T] \to \mathbf{R}$ such that the following conditions hold $dV_t d\mathbf{P}$-a.e.:*

(1) for all $v \in H^{\lambda+\mu}$,

$$\begin{aligned} 2(\Lambda^\mu v, \Lambda^{-\mu}\mathcal{L}_t v)_\lambda + |\mathcal{M}_t v|^2_{L_2(H^\lambda,\mathcal{H}^*_t)} &\le L|v|^2_\lambda; \\ 2(\Lambda^\mu v, \Lambda_t^{-\mu}\mathcal{L}_t v + f_t)_\lambda + |\mathcal{M}_t v + g_t|^2_{L_2(H^\lambda,\mathcal{H}^*_t)} &\le L|v|^2_\lambda + \bar f_t; \end{aligned}$$

(2) for all $v \in H^{\lambda+\mu}$,

$$|\mathcal{L}_t v|_{\lambda-\mu} \le K|v|_{\lambda+\mu}, \quad |\mathcal{M}_t v|_{L_2(H^\lambda,\mathcal{H}^*_t)} \le K|v|_{\lambda+\mu};$$

(3)

$$|f_t|^2_{\lambda-\mu} + |g_t|^2_{L_2(H^\lambda,\mathcal{H}^*_t)} \le \bar f_t, \quad \mathbf{E}\int_0^T \bar f_t dV_t < \infty.$$

Let O_T be the optional sigma-algebra on $\Omega \times [0,T]$. For $\mu \in (0,1])$ and $\lambda \in \mathbf{R}_+$ with $\lambda \in \{0,\alpha\}$, we denote by $\mathcal{W}^{\lambda,\mu}$ the space of all H^λ-valued strongly càdlàg processes $v : \Omega \times [0,T] \to H^\lambda$ that belong to $L^2(\Omega \times [0,T], O_T, dV_t d\mathbf{P}; H^{\lambda+\mu})$. The following is our definition of the solution of (4.2.2) and is standard in the variational theory or L^2-theory of stochastic evolution equations.

Definition 4.2.2. A process $u \in \mathcal{W}^{0,\mu}$ is said to be a solution of the stochastic evolution equation (4.2.2) if **P**-a.s. for all $t \in [0,T]$

$$u_t \overset{H^{-\mu}}{=} u_0 + \int_0^t (\mathcal{L}_s u_s + f_s) dV_s + \int_0^t (\mathcal{M}_s u_{s-} + g_s) dM_s,$$

where $\overset{H^{-\mu}}{=}$ indicates that the equality holds in the $H^{-\mu}$. That is, **P**-a.s. for all $t \in [0,T]$ and $v \in H^{\mu}$,

$$(v, u_t)_0 = (v, u_0) + \int_0^t \langle v, \mathcal{L}_s u_s + f_s \rangle_\mu dV_s + \int_0^t \{v(\mathcal{M}_s u_{s-} + g_s)\}_{H^0} dM_s.$$

Remark 4.2.3. In Definition 4.2.2, it is implicitly assumed that the integrals in (4.2.2) are well-defined. Moreover, it is easy to check that if Assumption 4.2.1$(0,\mu)$ holds, then the integrals in Definition 4.2.2 are well-defined.

In order to obtain estimates of the second moments of the supremum in t of the solution of (4.2.2), in the H^α norm, we will need to impose the upcoming assumption. Before introducing this assumption, we describe a few notational conventions. For two real-valued semimartingales X_t and Y_t, we write **P**-a.s. $dX_t \le dY_t$ if with probability 1, $X_t - X_s \le Y_t - Y_s$ for any $0 \le s \le t \le T$. For $v \in \mathcal{W}^{\lambda,\mu}$, we define

$$\mathfrak{M}_t(v) := \int_0^t \mathcal{M}_s v_s dM_s, \quad t \in [0,T],$$

and denote by $[\mathfrak{M}(v)]_{\lambda;t}$ the quadratic variation process of $\mathfrak{M}_t(v)$ in H^λ.

Assumption 4.2.2 (λ,μ)**.** *There is a positive constant L, a $\mathcal{P}_T$-measurable function $\bar{g} : \Omega \times [0,T] \to \mathbf{R}$, and increasing adapted processes $A, B : \Omega \times [0,T] \to \mathbf{R}$ with $dA_t d\mathbf{P} \le L dV_t d\mathbf{P}$, $dB_t d\mathbf{P} \le \bar{g}_t dV_t d\mathbf{P}$ on $\mathcal{P}_T$ such that the following conditions hold* **P**-*a.s.:*

(1) for all $v \in \mathcal{W}^{\lambda,\mu}$,

$$(\Lambda^\mu v_t, \Lambda^{-\mu} \mathcal{L}_t v_t)_\lambda dV_t + d[\mathfrak{M}(v)]_{\lambda;t} + 2\{v_{t-} \mathcal{M}_t v_{t-}\}_{H^\lambda} dM_t \le |v_{t-}|_\lambda^2 dA_t + G_t(v) dM_t,$$

where $G(v) \in \hat{L}^2_{loc}(Q)$ satisfies $|G_t(v)|_{\mathcal{H}_t^*} dV_t \le L |v_{t-}|_\lambda^2 dV_t$;

(2) for all $v \in \mathcal{W}^{\lambda,\mu}$,

$$2d[\mathfrak{M}(v), \mathcal{I}(g)]_{\lambda,t} + 2\{v_{t-} g_t\}_{H^\lambda} dM_t \le |v_{t-}|_\lambda dB_t + \tilde{G}_t(v) dM_t,$$

where $\bar{G}(v) \in \hat{L}^2_{loc}(Q)$ satisfies $|\bar{G}_t(v)|_{\mathcal{H}_t^*} dV_t \le L |v_{t-}|_\lambda \bar{g}_t dV_t$, and

$$\mathbf{E} \int_0^T \bar{g}_t^2 dV_t < \infty.$$

Although Assumption 4.2.2(λ,μ) looks rather technical, it is satisfied for a large class of parabolic stochastic integro-differential equations (see Section 4.3) under what we consider to be reasonable assumptions.

Let $\mathcal{T}$ be the set of all stopping times $\tau \le T$ and $\mathcal{T}^p$ be the set of all predictable stopping times $\tau \le T$.

Theorem 4.2.4. *Let $\mu \in (0,1]$ and $\alpha \ge \mu$. Let Assumption 4.2.1(λ,μ) hold for $\lambda \in \{0,\alpha\}$ and assume that $\mathbf{E}\left[|\varphi|_\alpha^2\right] < \infty$.*

(1) Then there exists a unique solution $u = (u_t)_{t\le T}$ of (4.2.2) such that for any $\alpha' < \alpha$, u is an $H^{\alpha'}$-valued strongly càdlàg process and there is a constant $N = N(L,K,C)$ such that

$$\mathbf{E}\left[\sup_{t\le T}|u_t|_{\alpha-\mu}^2\right] + \sup_{\tau\in\mathcal{T}}\mathbf{E}\left[|u_\tau|_\alpha^2\right] + \mathbf{E}\int_0^T |u_s|_\alpha^2\,dV_s \le N\left(\mathbf{E}\left[|\varphi|_\alpha^2\right] + \mathbf{E}\int_0^T \bar{f}_t dV_t\right).$$

Moreover, for all $p \in (0,2)$ and $\alpha' < \alpha$, there is a constant $N = N(L,K,C,p,\alpha')$ such that

$$\mathbf{E}\left[\sup_{t\le T}|u_t|_{\alpha'}^p\right] \le N\mathbf{E}\left[\left(|\varphi|_\alpha^2 + \int_0^T \bar{f}_t dV_t\right)^{\frac{p}{2}}\right].$$

(2) If, in addition, Assumption 4.2.2(λ,μ) holds for $\lambda \in \{0,\alpha\}$, then u is an H^α-valued weakly càdlàg process and there is a constant $N = N(L,K,C)$ such that

$$\mathbf{E}\left[\sup_{t\le T}|u_t|_\alpha^2\right] \le N\mathbf{E}\left[|\varphi|_\alpha^2 + \int_0^T (\bar{f}_t + \bar{g}_t^2)dV_t\right].$$

Remark 4.2.5. If V is an arbitrary continuous increasing adapted process, then Theorem 4.2.4 can be applied locally by considering $V_t^C = V_{t\wedge\tau_C}, t \in [0,T]$, with $\tau_C = \inf(t \in [0,T] : V_t \ge C) \wedge T$.

4.2.3 Proof of Theorem 4.2.4

We will construct a sequence of approximations in $\mathcal{W}_T^{\alpha,\mu}$ of the solution of (4.2.2) by solving in the triple $(H^{-\mu}, H^0, H^\mu)$ the equation

$$\begin{aligned} du_t &= (\mathcal{L}_t^n u_t + f_t)\,dV_t + (\mathcal{M}_t u_{t-} + g_t)\,dM_t, \quad t \le T, \\ u_0 &= \varphi, \end{aligned} \tag{4.2.3}$$

where $\mathcal{L}_t^n = \mathcal{L}_t - \frac{1}{n}(\Lambda^\mu)^2$. In order to apply the foundational theorems on stochastic evolution equations with jumps established in [GK81] and [Gyö82], it is convenient for us first to consider the following equation in the triple $(H^{-\mu}, H^0, H^\mu)$:

$$
\begin{aligned}
dv_t &= \Big(\Lambda^\alpha \mathcal{L}_t \Lambda^{-\alpha} v_t - \frac{1}{n}(\Lambda^\mu)^2 v_t + \Lambda^\alpha f_t\Big)\, dV_t + \big(\mathcal{M}_t \Lambda^{-\alpha} v_{t-}(\Lambda^\alpha)^* + g_t(\Lambda^\alpha)^*\big)\, dM_t, \quad t \le T, \\
v_0 &= \Lambda^\alpha \varphi.
\end{aligned} \tag{4.2.4}
$$

The solutions of (4.2.3) and (4.2.4) are to be understood following Definition 4.2.2.

Lemma 4.2.6. *Let $\mu \in (0,1]$ and $\alpha \ge \mu$. Let Assumption 4.2.1(α,μ) hold and assume that $\mathbf{E}\left[|\varphi|_\alpha^2\right] < \infty$.*

(1) For each $n \in \mathbf{N}$, there is a unique solution $v^n = (v^n_t)_{t\le T}$ of (4.2.3), and there is a constant $N = N(L,K,C)$ independent of n such that

$$
\sup_{\tau\in\mathcal{T}} \mathbf{E}\left[|v^n_\tau|_0^2\right] + \mathbf{E}\int_0^T |v^n_t|_0^2 dV_t + \frac{1}{n}\mathbf{E}\int_0^T |v^n_t|_\mu^2 dV_t \le N\mathbf{E}\left[|\varphi|_\alpha^2 + \int_0^T \bar{f}_t dV_t\right]. \tag{4.2.5}
$$

Moreover, for all $p \in (0,2)$, there is a constant $N = N(L,K,T,p)$

$$
\mathbf{E}\left[\sup_{t\le T} |v^n_t|_0^p\right] \le N\mathbf{E}\left[\left(|\varphi|_\alpha^2 + \int_0^T \bar{f}_t dV_t\right)^{\frac{p}{2}}\right]. \tag{4.2.6}
$$

(2) If, in addition, Assumption 4.2.2(α,μ) holds, then there is a constant $N = N(L,K,C)$ such that

$$
\mathbf{E}\left[\sup_{t\le T} |v^n_t|_0^2\right] \le N\mathbf{E}\left[|\varphi|_\alpha^2 + \int_0^T (\bar{f}_t + \bar{g}_t^2) dV_t\right]. \tag{4.2.7}
$$

Proof. (1) For each $(\omega,t) \in \Omega\times[0,T]$ and $n \in \mathbf{N}$, let

$$
\mathcal{L}^\alpha_t v = \Lambda^\alpha \mathcal{L}_t \Lambda^{-\alpha} v, \quad \mathcal{L}^{\alpha,n}_t v = \mathcal{L}^\alpha_t v - \frac{1}{n}(\Lambda^\mu)^2 v, \quad \mathcal{M}^\alpha_t v = \mathcal{M}_t \Lambda^{-\alpha} v(\Lambda^\alpha)^*.
$$

Using basic properties of the spaces $(H^\alpha)_{\alpha\in\mathbf{R}}$ and the operators $(\Lambda^\alpha)_{\alpha\in\mathbf{R}}$, $dV_t d\mathbf{P}$-a.e. for all $v \in H^\mu$, we have

$$
2\langle v, \mathcal{L}^\alpha_t v\rangle_\mu = 2(\Lambda^\mu v, \Lambda^{-\mu}\Lambda^\alpha \mathcal{L}_t \Lambda^{-\alpha} v)_0 = 2(\Lambda^\mu \Lambda^{-\alpha} v, \Lambda^{-\mu}\mathcal{L}_t \Lambda^{-\alpha} v)_\alpha,
$$

$$
2\langle v, (\Lambda^\mu)^2 v\rangle_\mu = 2(\Lambda^\mu v, \Lambda^\mu v)_0 = |v|_\mu^2,
$$

and

$$
\begin{aligned}
\left|\mathcal{M}^\alpha_t v\right|^2_{L_2(H^0,\mathcal{H}^*_t)} &= \sum_{k=1}^\infty |\Lambda^\alpha(\mathcal{M}_t\Lambda^{-\alpha} v)^* \tilde{e}^n_t|^2_{H^0} = \sum_{k=1}^\infty |(\mathcal{M}_t\Lambda^{-\alpha} v)^* \tilde{e}^n_t|^2_{H^\alpha} \\
&= \sum_{k=1}^\infty |\mathcal{M}_t\Lambda^{-\alpha} v\bar{h}^k|^2_{\mathcal{H}^*_t} = \left|\mathcal{M}^\alpha_t v\right|^2_{L_2(H^\alpha,\mathcal{H}^*_t)},
\end{aligned}
$$

where $(\tilde{e}_t^k)_{k\in\mathbf{N}}$,and $(\bar{h}^k)_{k\in\mathbf{N}}$ are orthonormal basis of $\mathcal{H}_t$ and H^α, respectively. It follows from Assumption 4.2.1(α,μ) that $dV_t d\mathbf{P}$-a.e. for all $v\in H^\mu$, we have

$$|\mathcal{L}_t^{\alpha,n}v|_{-\mu}\le\left(K+\frac{1}{n}\right)|v|_\mu,\quad \left|\mathcal{M}_t^\alpha v\right|_{L_2(H^0,\mathcal{H}_t^*)}\le K|v|_\mu,$$

and

$$2\langle v,\mathcal{L}_t^{\alpha,n}v+\Lambda^\alpha f_t\rangle_\mu+\left|\mathcal{M}_t^\alpha v+g_t(\Lambda^\alpha)^*\right|^2_{L_2(H^0,\mathcal{H}_t^*)}\le-\frac{2}{n}|v|_\mu^2+L|v|_0^2+\bar{f}_t. \tag{4.2.8}$$

In [Gyö82], the variational theory for monotone stochastic evolution equations driven by locally square integrable Hilbert-space-valued martingales was derived; it is worth mentioning that the càdlàg version of the variational solution in the pivot space and the uniqueness of the solution was obtained using Theorem 2 in [GK81]. The theorems and proofs given in [Gyö82] continue to hold for equations driven by the cylindrical martingales we consider in this paper. Therefore, by Theorems 2.9 and 2.10 in [Gyö82], for every $n\in\mathbf{N}$, there exists a unique solution $v^n=(v_t^n)_{t\le T}$ of the stochastic evolution equation given by

$$v_t^n=\Lambda^\alpha\varphi+\int_0^t(\mathcal{L}_s^{\alpha,n}v_s^n+\Lambda^\alpha f_s)dV_s+\int_0^t(\mathcal{M}_s^\alpha v_{s-}^n+\Lambda^\alpha g_s)dM_s,\quad t\le T.$$

Furthermore, by virtue of Theorem 4.1 in [Gyö82], there is a constant $N(n)=N(n,L,K,C)$ such that

$$\mathbf{E}\left[\sup_{t\le T}|v_t^n|_0^2\right]+\mathbf{E}\int_0^T|v_t^n|_\mu^2dV_t\le N(n)\mathbf{E}\left[|\varphi|_\alpha^2+\int_0^T\bar{f}_tdV_t\right]. \tag{4.2.9}$$

We will now use our assumptions to obtain bounds of the solutions v_t^n, $n\in\mathbf{N}$, in the H^0-norm independent of n. Applying Theorem 2 in [GK81], $\mathbf{P}$-a.s. for all $t\in[0,T]$, we have

$$|v_t^n|_0^2=|\varphi|_\alpha^2+\int_0^t 2\langle v_s^n,\mathcal{L}_t^{\alpha,n}v_s^n+\Lambda^\alpha f_s\rangle_\mu dV_s+\left[\tilde{\mathfrak{M}}\right]_{0,t}+m_t, \tag{4.2.10}$$

where $(\tilde{\mathfrak{M}}_t)_{t\le T}$ and $(m_t)_{t\le T}$ are local martingales given by

$$\tilde{\mathfrak{M}}_t:=\int_0^t(\mathcal{M}_s^\alpha v_{s-}^n+\Lambda^\alpha g_s)dM_s,\quad m_t:=2\int_0^t\{v_{s-}^n(\mathcal{M}_s^\alpha v_{s-}^n+\Lambda^\alpha g_s)\}_{H^0}dM_s.$$

Thus, taking the expectation of (4.2.10) and making use of Assumption 4.2.1(α,μ), (4.2.8), and (4.2.9), we find that for all $\tau\in\mathcal{T}$,

$$\begin{aligned}\mathbf{E}\left[|v_\tau^n|_0^2\right]&\le\mathbf{E}\left[|\varphi|_\alpha^2\right]+\mathbf{E}\int_0^\tau\left(2\langle v_t^n,\mathcal{L}_t^{\alpha,n}v_t^n+\Lambda^\alpha f_t\rangle_\mu+|\mathcal{M}_t^\alpha v+\Lambda^\alpha g_t|^2_{L_2(H^0,\mathcal{H}_t)}\right)dV_t\\&\le\mathbf{E}\left[|\varphi|_\alpha^2+\int_0^\tau\left(L|v_t^n|_0^2-\frac{2}{n}|v_t^n|_\mu^2+\bar{f}_t\right)dV_t\right].\end{aligned}$$

This implies that for any $\tau \in \mathcal{T}^p$,

$$\mathbf{E}\left[|v^n_{\tau-}|^2_0\right] + \frac{2}{n}\mathbf{E}\int_0^\tau |v^n_t|^2_\mu dV_t \le \mathbf{E}\left[|\varphi|^2_\alpha + \int_0^\tau \left(L|v^n_s|^2_0 + \bar{f}_s\right)dV_s\right].$$

By virtue of Lemmas 2 and 3 in [GM83], we deduce that there is a constant $N = N(L, K, C)$ such that for any $\tau \in \mathcal{T}$,

$$\mathbf{E}\left[|v^n_\tau|^2_0\right] + \frac{1}{n}\mathbf{E}\int_0^\tau |v^n_t|^2_\mu dV_t \le N\mathbf{E}\left[|\varphi|^2_\alpha + \int_0^\tau \bar{f}_s dV_s\right],$$

which implies that (4.2.5) holds since V_t is uniformly bounded by the constant C. Finally, owing to Corollary II in [Len77], we have (4.2.6).

(2) Using Assumption 4.2.2(α, μ) and estimating (4.2.10), we get that $\mathbf{P}$-a.s.

$$d|v^n_t|^2_0 \le |\varphi|^2_\alpha + |v^n_{t-}|^2_\lambda dA_t + |v^n_{t-}|_\lambda dB_t + (G_t(v^n) + \bar{G}_t(v^n))dM_t.$$

Moreover, for any $\tau \in \mathcal{T}$, we obtain

$$\mathbf{E}\left[\sup_{t\le\tau}|v^n_t|^2_0\right] \le N\mathbf{E}\left[|\varphi|^2_\alpha + \int_0^\tau |v^n_t|^2_0 dV_t + \int_0^\tau (\bar{f}_t + \bar{g}^2_t)dV_t + \sup_{t\le\tau}|l^n_t|\right],$$

where $l^n_t := \int_0^t (G_s(v^n) + \bar{G}_s(v^n))dM_s$. Moreover, by the Burkholder-Davis-Gundy inequality and Young's inequality, we have

$$\begin{aligned}\mathbf{E}\sup_{t\le\tau}|l^n_t| &\le N\mathbf{E}\left[\left(\int_0^\tau (|v^n_{t-}|^4_\alpha + |v^n_{t-}|^2_\alpha \bar{g}^2_t)dV_t\right)^{\frac{1}{2}}\right] \le N\mathbf{E}\left[\sup_{t\le\tau}|v^n_t|_0\left(\int_0^\tau (|v^n_{t-}|^2_\alpha + \bar{g}^2_t)dV_t\right)^{\frac{1}{2}}\right]\\ &\le \frac{1}{4N}\mathbf{E}\left[\sup_{t\le\tau}|v^n_t|^2_0\right] + N\mathbf{E}\int_0^\tau (|v^n_t|^2_\alpha + \bar{g}^2_t)dV_t,\end{aligned}$$

from which estimate (4.2.7) follows. □

Proof of Theorem 4.2.4 . (1) Let $v^n = (v^n_t)_{t\le T}$ be the unique solution of (4.2.4) constructed in Lemma 4.2.6. Since $v^n \in \mathcal{W}^{0,\mu}$, it follows that $u^n := \Lambda^{-\alpha}v^n \in \mathcal{W}^{\alpha,\mu} \subseteq \mathcal{W}^{0,\mu}$ is a solution of (4.2.3) in the triple $(H^{-\mu}, H^0, H^\mu)$.

We will first show that $(u^n)_{n\in\mathbf{N}}$ is Cauchy in H^0. Letting $u^{n,m} = u^n - u^m$, for all $n, m \in \mathbf{N}$, we have

$$u^{n,m}_t = \int_0^t (\mathcal{L}^n_t u^n_t - \mathcal{L}^m_t u^m_t)dV_t + \int_0^t \mathcal{M}_t u^{n,m} dM_t, \quad t \le T.$$

Applying Theorem 2 in [GK81], we obtain that $\mathbf{P}$-a.s. for all $t \in [0, T]$,

$$|u^n_t - u^m_t|^2_0 = \int_0^t 2\langle u^n_s - u^m_s, \mathcal{L}^n_s u^n_s - \mathcal{L}^m_s u^m_s\rangle_{\mu,0} dV_s + [\mathfrak{M}^{n,m}]_t + \eta^{n,m}_t \tag{4.2.11}$$

where $(\mathfrak{M}^{n,m}_t)_{t\leq T}$ and $(\eta^{n,m}_t)_{t\leq T}$ are local martingales given by

$$\mathfrak{M}^{n,m}_t := \int_0^t \mathcal{M}_s(u^n_{s-} - u^m_{s-})dM_s, \quad \eta^{m,n}_t := 2\int_0^t \{(u^n_{s-} - u^m_{s-})\mathcal{M}_s(u^n_{s-} - u^m_{s-})\}_{H^0} dM_s.$$

Assumption 4.2.1$(0,\mu)$ (1) implies that for any $\tau \in \mathcal{T}$,

$$\begin{aligned}\mathbf{E}\left[|u^{n,m}_\tau|_0^2\right] &\leq \mathbf{E}\int_0^\tau \left(2\langle u^{n,m}_s, \mathcal{L}^n_s u^{n,m}_s\rangle_\mu + |\mathcal{M}_t u^{n,m}_s|^2_{L_2(H^0,\mathcal{H}^*_t)}\right) dV_s \\ &\leq \mathbf{E}\int_0^\tau \left(L|u^{n,m}_s|_0^2 + \frac{1}{n}|u^n_s|_\mu^2 + \frac{1}{m}|u^m_s|_\mu^2\right) dV_s,\end{aligned}$$

and hence for any $\tau \in \mathcal{T}^p$, we have

$$\mathbf{E}\left[|u^{n.m}_{\tau-}|_0^2\right] \leq \mathbf{E}\int_0^\tau \left(L|u^{n.m}_s|_0^2 + \frac{1}{n}|u^n_s|_\mu^2 + \frac{1}{m}|u^m_s|_\mu^2\right) dV_s.$$

By virtue of Lemmas 2 and 4 in [GM83] and (4.2.5) (noting that $|u^n_t|_0 = |\Lambda^{-\alpha} v^n_t|_0 \leq N|v^n_t|_0$), there is a constant N such that for any $\tau \in \mathcal{T}$,

$$\mathbf{E}\left[|u^{n,m}_\tau|_0^2\right] \leq N\mathbf{E}\int_0^\tau \left(\frac{1}{n}|u^n_s|_\mu^2 + \frac{1}{m}|u^m_s|_\mu^2\right) dV_s \leq N\left(\frac{1}{n} + \frac{1}{m}\right)\mathbf{E}\left[|\varphi|_\alpha^2 + \int_0^T \bar{f}_t dV_t\right].$$

Using Corollary II in [Len77], we have that for all $p \in (0,2)$, there is a constant N such that

$$\mathbf{E}\left[\sup_{t\leq T} |u^{n,m}_t|_0^p\right] \leq N\left(\frac{1}{n} + \frac{1}{m}\right)^{\frac{p}{2}} \left[\mathbf{E}\left[|\varphi|_\alpha^2 + \int_0^T \bar{f}_t dV_t\right]\right]^{\frac{p}{2}}.$$

Therefore,

$$\lim_{n,m\to\infty} \left[\sup_{\tau\in\mathcal{T}} \mathbf{E}\left[|u^{n,m}_\tau|_0^2\right] + \mathbf{E}\left[\sup_{t\leq T} |u^{n,m}_t|_0^p\right]\right] = 0, \tag{4.2.12}$$

and hence there exists a strongly càdlàg H^0-valued process $u = (u_t)_{t\leq T}$ such that

$$d\mathbf{P} - \lim_{n\to\infty} \sup_{t\leq T} |u_t - u^n_t|_0 = 0. \tag{4.2.13}$$

Since for all n, u^n is solution of (4.2.3), we have that $\mathbf{P}$-a.s. for all $t \in [0,T]$ and $\phi \in H^\mu$,

$$(\phi, u^n_t)_0 = (\phi,\varphi)_0 + \int_0^t \langle \phi, \mathcal{L}^n_s u^n_s + f_s\rangle_{\mu,0} dV_s + \int_0^t \{\phi(\mathcal{M}_s u^n_{s-} + g_s)\}_{H^0} dM_s. \tag{4.2.14}$$

Owing to (4.2.13), we know that the left-hand-side of (4.2.14) converges $\mathbf{P}$-a.s. for all $t \in [0,T]$ to $(\phi, u_t)_0$ as n tends to infinity. Our aim, of course, is to pass to the limit as n tends to infinity on the right-hand-side.

This can be done quite simply when $\alpha > \mu$. Indeed, owing to the interpolation inequality (4.2.1), for all $\varepsilon > 0$, $\alpha' < \alpha$, and $p \in (0,2)$, there is a constant $N = N(\alpha, \alpha', \varepsilon, p)$ such that

$$\sup_{t\le T} |u_t^{n,m}|_{\alpha'}^p \le \varepsilon \sup_{t\le T} |u_t^{n,m}|_{\alpha}^p + N \sup_{t\le T} |u_t^{n,m}|_0^p. \tag{4.2.15}$$

Since $|u_t^n|_\alpha = |\Lambda^{-\alpha} v_t^n|_\alpha \le N|v_t^n|_0$, by (4.2.7) and (4.2.15), we have that for all $\alpha' < \alpha$ and $p \in (0,2)$,

$$\mathbf{E}\left[\sup_{t\le T} |u_t^{n,m}|_{\alpha'}^p\right] \le \varepsilon N \mathbf{E}\left[\left(|\varphi|_\alpha^2 + \int_0^T \bar{f}_t dV_t\right)^{\frac{p}{2}}\right] + N\mathbf{E}\left[\sup_{t\le T} |u_t^{n,m}|_0^p\right]. \tag{4.2.16}$$

Using (4.2.12) and passing to the limit as n and m tend to infinity on both sides of (4.2.16), and then taking $\varepsilon \downarrow 0$, we get that for all $\alpha' < \alpha$ and $p \in (0,2)$,

$$\lim_{n,m\to\infty} \mathbf{E}\left[\sup_{t\le T} |u_t^{n,m}|_{\alpha'}^p\right] = 0.$$

Combining the above results, we conclude that for any $\alpha' < \alpha$, u is an $H^{\alpha'}$-valued strongly càdlàg process and

$$d\mathbf{P} - \lim_{n\to\infty} \sup_{t\le T} |u_t - u_t^n|_{\alpha'} = 0. \tag{4.2.17}$$

In particular, if $\alpha > \mu$, then taking $\alpha' > \mu$ in (4.2.17) and appealing to Assumption 4.2.1$(0,\mu)$ (2), (4.2.5), and the identity,

$$\langle \phi, \Lambda^{2\mu} u_s^n \rangle_\mu = (\Lambda^\mu \phi, \Lambda^\mu u_s^n)_0,$$

we can take the limit as n tends to infinity on the right-hand-side of (4.2.14) by the dominated convergence theorem to conclude that u is a solution of (4.2.2).

The case $\alpha = \mu$ must be handled with weak convergence. Let

$$S(\mathcal{O}_T) = (\Omega \times [0,T], \mathcal{O}_T, d\bar{V}_t d\mathbf{P}) \quad \textit{and} \quad S(\mathcal{P}_T) = (\Omega \times [0,T], \mathcal{P}_T, d\bar{V}_t d\mathbf{P}),$$

where $\bar{V}_t =: V_t + t$. It follows from (4.2.5) that there exists a subsequence $(u^{n_k})_{k\in\mathbf{N}}$ of $(u^n)_{n\in\mathbf{N}}$ that converges weakly in $L^2(S(\mathcal{O}_T); H^\mu)$ to some $\bar{u} \in L^2(S(\mathcal{O}_T); H^\mu)$ which satisfies

$$\mathbf{E}\int_{]0,T]} |\bar{u}_t|_\mu^2 d\bar{V}_t \le N\mathbf{E}\left[|\varphi|_\mu^2 + \int_0^T \bar{f}_t dV_t\right].$$

For any $\phi \in H^0$ and bounded predictable process ξ_t, we have

$$\lim_{k\to\infty} \mathbf{E}\int_0^T \xi_t \langle \phi, u_t^{n_k}\rangle_\mu d\bar{V}_t = \lim_{k\to\infty} \mathbf{E}\int_0^T \xi_t (u_t^{n_k}, \phi)_0 d\bar{V}_t = \mathbf{E}\int_0^T \xi_t (u_t, \phi)_0 d\bar{V}_t,$$

and hence $u = \bar{u}$ in $L^2(S(O_T); H^\mu)$ and u^{n_k} converges to u weakly in $L^2(S(O_T); H^\mu)$ as k tends to infinity. Define $u_-^{n_k} = (u_{t-}^{n_k})_{t\le T}$ and $u_- = (u_{t-})_{t\le T}$, where the limits are taken in H^0. By repeating the argument above, we conclude that $u_-^{n_k}$ converges to u_- weakly in $L^2(S(O_T); H^\mu)$ as k tends to infinity.

Fix $\phi \in H^\mu$ and a $\mathcal{P}_T$-measurable process $(\xi_t)_{t\le T}$ bounded by the constant K. Define the linear functionals $\Phi^{\mathcal{L}} : L^2(S(O_T); H^\mu) \to \mathbf{R}$ and $\Phi^{\mathcal{M}} : L^2(S(\mathcal{P}_T); H^\mu) \to \mathbf{R}$ by

$$\Phi^{\mathcal{L}}(v) = \mathbf{E}\int_{]0,T]} \xi_t \int_{]0,t]} \langle \phi, \mathcal{L}_s v_s\rangle_{\mu,0} dV_s d\bar{V}_t, \quad \forall v \in L^2(S(O_T); H^\mu)$$

and

$$\Phi^{\mathcal{M}}(v) = \mathbf{E}\int_{]0,T]} \xi_t \int_{]0,t]} \{\phi \mathcal{M}_s v_s\}_{H^0} dM_s d\bar{V}_t, \quad \forall v \in L^2(S(\mathcal{P}_T); H^\mu),$$

respectively. Owing to Assumption 4.2.1$(0,\mu)$ (2), the Burkholder-Davis-Gundy inequality, and the fact that $(\bar{V}_t)_{t\le T}$ is uniformly bounded by the constant C, there is a constant $N = N(K, C)$ such that

$$|\Phi^{\mathcal{L}}(v)| \le N|\phi|_\mu \left(\mathbf{E}\int_0^T |v_t|_\mu^2 d\bar{V}_t\right)^{\frac{1}{2}}, \quad \forall v \in L^2(S(O_T); H^\mu),$$

and

$$|\Phi^{\mathcal{M}}(v)| \le N|\phi|_\mu \left(\mathbf{E}\int_{]0,T]} |v_s|_\mu^2 d\bar{V}_s\right)^{\frac{1}{2}}, \quad \forall v \in L^2(S(\mathcal{P}_T); H^\mu).$$

This implies that $\Phi^{\mathcal{L}}$ is a continuous linear functional on $L^2(S(O_T); H^\mu)$ and $\Phi^{\mathcal{M}}$ is a continuous linear functional on $L^2(S(\mathcal{P}_T); H^\mu)$, and hence that

$$\lim_{k\to\infty} \Phi^{\mathcal{L}}(u^{n_k}) = \Phi^{\mathcal{L}}(u), \quad \lim_{k\to\infty} \Phi^{\mathcal{L}}(u_-^{n_k}) = \Phi^{\mathcal{M}}(u_-). \tag{4.2.18}$$

For each k, we have that

$$\mathbf{E}\int_0^T \xi_t(\phi, u_t^{n_k})_0 d\bar{V}_t = \mathbf{E}\int_0^T \xi_t(\phi, \varphi)_0 d\bar{V}_t + \mathbf{E}\int_0^T \xi_t \int_0^t \langle \phi, \mathcal{L}_s^{n_k} u_s^{n_k} + f_s\rangle_{\mu,0} dV_s d\bar{V}_t \tag{4.2.19}$$
$$+\mathbf{E}\int_0^T \xi_t \int_0^t \{\phi(\mathcal{M}_s u_{s-}^n + g_s)\}_{H^0} dM_s d\bar{V}_t.$$

Passing to the limit as k tends to infinity on both sides of (4.2.19) using (4.2.18) and

$$\langle \phi, \Lambda^{2\mu} u_s^{n_k}\rangle_\mu = (\Lambda^\mu \phi, \Lambda^\mu u_s^{n_k})_0,$$

we obtain that $d\bar{V}_t d\mathbf{P}$-a.e.

$$(\phi, u_t)_0 = (\phi, \varphi)_0 + \int_{]0,t]} \langle \phi, (\mathcal{L}_s u_s + f_s) \rangle_\mu dV_s + \int_{]0,t]} \{\phi(\mathcal{M}_s u_{s-} + g_s)\}_{H_0} dM_s, \quad t \le T.$$

Therefore, for all $\alpha \ge \mu$, u is a solution of (4.2.2).

We will now show that

$$\sup_{\tau \in \mathcal{T}} \mathbf{E}\left[|u_\tau|_\alpha^2\right] \le N\mathbf{E}\left[|\varphi|_\alpha^2 + \int_0^T \bar{f}_t dV_t\right]. \tag{4.2.20}$$

Let $(h^k)_{k\in\mathbf{N}}$ be a complete orthonormal basis in H^α such that the linear subspace spanned by $(h^k)_{k\in\mathbf{N}}$ is dense in $H^{2\alpha}$. Owing to (4.2.5), for all $m \ge 1$ and $\tau \in \mathcal{T}$,

$$\mathbf{E}\left[\sum_{k=1}^m |(u_\tau^n, \Lambda^{2\alpha} h^k)_0|^2\right] = \mathbf{E}\left[\sum_{k=1}^m |(u_\tau^n, h^k)_\alpha|^2\right] \le \mathbf{E}\left[|u_\tau^n|_\alpha^2\right] \le N\mathbf{E}\left[|\varphi|_\alpha^2 + \int_0^T \bar{f}_t dV_t\right].$$

Applying Fatou's lemma first in n and then in m, we have that for all $\tau \in \mathcal{T}$,

$$\mathbf{E}\left[\sum_{k=1}^\infty |(u_\tau, \Lambda^{2\alpha} h^k)_0|^2\right] \le N\mathbf{E}\left[|\varphi|_\alpha^2 + \int_0^T \bar{f}_t dV_t\right].$$

Hence, for all $t \in [0,T]$, $\mathbf{P}$-a.s. $v_t = \sum_k (u_t, \Lambda^{2\alpha} h^k)_0 h^k \in H^\alpha$. Since the linear subspace spanned by $(\Lambda^{2\alpha} h^k)_{k\in\mathbf{N}}$ is dense in H^0 and for all $t \in [0,T]$, $\mathbf{P}$-a.s., $(u_t - v_t, \Lambda^{2\alpha} h^k)_0 = 0$, for all $k \in \mathbf{N}$, it follows that $\mathbf{P}$-a.s. for all $\tau \in \mathcal{T}$, $u_\tau = v$ and

$$\mathbf{E}\left[|u_\tau|_\alpha^2\right] = \mathbf{E}\sum_{k=1}^\infty \left|(u_\tau, \Lambda^{2\alpha} h^k)_0\right|^2 \le N\mathbf{E}\left[|\varphi|_\alpha^2 + \int_0^T \bar{f}_t dV_t\right],$$

which proves (4.2.20).

Estimating (4.2.2) directly in the $H^{\alpha-\mu}$-norm, we easily derive that

$$\mathbf{E}\left[\sup_{t\le T} |u_t|_{\alpha-\mu}^2\right] \le N\mathbf{E}\left[|\varphi|_\alpha^2 + \int_0^T \bar{f}_s dV_t\right].$$

If $(v_t)_{t\le T}$ be another solution of (4.2.2), then by Theorem 2 in [GK81] and Assumption 4.2.1$(0,\mu)(1)$, $\mathbf{P}$-a.s. for all $t \in [0,T]$, we have

$$|u_t - v_t|_0^2 \le L\int_0^t |u_s - v_s|_0^2 dV_s + m_t,$$

where $(m_t)_{t\le T}$ is a local martingale with $m_0 = 0$, and hence applying Lemmas 2 and 4 in

[GM83], we get

$$\mathbf{P}\left(\sup_{t\le T}|u_t - v_t|_0 > 0\right) = 0,$$

which implies that $(u_t)_{t\le T}$ is the unique solution of (4.2.2). This completes the proof of part (1).

(2) Estimating (4.2.11) using Assumption 4.2.2$(0,\mu)$, we get that $\mathbf{P}$-a.s.

$$d|u_t^{n,m}|_0^2 \le |\varphi|_0^2 + \left(\frac{1}{n}+\frac{1}{m}\right)|u^{n,m}|_0^2 dV_t + |u_{t-}^{n,m}|_\lambda^2 dA_t + |u_{t-}^{n,m}|_\lambda dB_t + (G_t(u^{n,m}) + \bar{G}_t(u^{n,m}))dM_t.$$

Then estimating the stochastic integrand as in the proof of part (2) of Lemma 4.2.6, for any $\tau \in \mathcal{T}$, we get

$$\mathbf{E}\left[\sup_{t\le\tau}|u_t^{n.m}|_0^2\right] \le N\mathbf{E}\int_0^\tau \left(|u_s^{n.m}|_0^2 + \frac{1}{n}|u_s^n|_\mu^2 + \frac{1}{m}|u_s^m|_\mu^2)\right) dV_s,$$

and hence by Gronwall's lemma and Lemma 4.2.6(2),

$$\mathbf{E}\left[\sup_{t\le\tau}|u_t^{n.m}|_0^2\right] \le N\left(\frac{1}{n}+\frac{1}{m}\right)\mathbf{E}\left[|\varphi|_\alpha^2 + \int_{]0,T]}\left(\bar{f}_t + \bar{g}_t^2\right)dV_t\right]$$

Thus,

$$\lim_{n,m\to\infty}\mathbf{E}\left[\sup_{t\le\tau}|u_t^{n.m}|_0^2\right] = 0.$$

Let $(h^k)_{k\in\mathbf{N}}$ be a complete orthonormal basis H^α such that the linear subspace spanned by $(h^k)_{k\in\mathbf{N}}$ is dense in $H^{2\alpha}$. Owing to (4.2.7), for all $m \ge 1$ and $\tau \in \mathcal{T}$,

$$\begin{aligned}\mathbf{E}\left[\sup_{t\le T}\sum_{k=1}^m (u_t^n, h^k)_\alpha\right] = \mathbf{E}\left[\sup_{t\le T}\sum_{k=1}^m |(u_t^n, \Lambda^{2\alpha}h^k)_0|^2\right] &\le \mathbf{E}\left[\sup_{t\le T}|u_t^n|_\alpha^2\right]\\ &\le N\mathbf{E}\left[|\varphi|_\alpha^2 + \int_{]0,T]}(\bar{f}_t + \bar{g}_t^2)dV_t\right].\end{aligned}$$

Applying Fatou's lemma first in n and then in m, we have that

$$\mathbf{E}\left[\sup_{t\le T}\sum_{k=1}^\infty |(u_t, \Lambda^{2\alpha}h^k)_0|^2\right] \le N\mathbf{E}\left[|\varphi|_\alpha^2 + \int_{]0,T]}(\bar{f}_t + \bar{g}_t^2)dV_t\right].$$

Thus, $v = \sum_k (u, \Lambda^{2\alpha}h^k)_0 h^k$ is an H^α-valued weakly càdlàg process. Since the linear subspace spanned on $(\Lambda^{2\alpha}h^k)_{k\in\mathbf{N}}$ is dense in H^0 and $\left(u_t - v_t, \Lambda^{2\alpha}h^k\right)_0 = 0$, for all $k \in \mathbf{N}$, it follows that $\mathbf{P}$-a.s. for all $t \in [0,T]$, $u_t = v_t$ and

$$\mathbf{E}\left[\sup_{t\le T}|u_t|_\alpha^2\right] = \mathbf{E}\left[\sup_{t\le T}\sum_{k=1}^\infty |(u_t, \Lambda^{2\alpha}h^k)_0|^2\right] \le N\mathbf{E}\left[|\varphi|_\alpha^2 + \int_{]0,T]}(\bar{f}_t + \bar{g}_t^2)dV_t\right].$$

$\square$

4.3 The L^2-Sobolev theory for degenerate SIDEs

4.3.1 Statement of main results

In this section, we consider the d_2-dimensional system of SIDEs on $[0,T]\times\mathbf{R}^{d_1}$ given by

$$\begin{aligned} du^l_t &= \left((\mathcal{L}^{1;l}_t+\mathcal{L}^{2;l}_t)u_t + b^i_t\partial_i u^l_t + c^{l\bar{l}}_t u^{\bar{l}}_t + f^l_t\right)dV_t + \left(\mathcal{N}^{l\varrho}_t u_t + g^{l\varrho}_t\right)dw^\varrho_t \\ &\quad + \int_{Z^1}\left(\mathcal{I}^l_{t,z}u^{\bar{l}}_{t-} + h^l_t(z)\right)\tilde{\eta}(dt,dz), \\ u^l_0 &= \varphi^l, \quad l\in\{1,\dots,d_2\}, \end{aligned} \tag{4.3.1}$$

where for $k\in\{1,2\}$, $l\in\{1,\dots,d_2\}$, and $\phi\in C^\infty_c(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$,

$$\begin{aligned} \mathcal{L}^{k;l}_t\phi(x) &:= \frac{1}{2}\sigma^{k;i\varrho}_t(x)\sigma^{k;j\varrho}_t(x)\partial_{ij}\phi^l(x) + \sigma^{k;i\varrho}_t(x)\upsilon^{k;l\bar{l}\varrho}_t(x)\partial_i\phi^{\bar{l}}(x) \\ &\quad + \int_{Z^k}\left(\left(\delta_{l\bar{l}}+\rho^{k;l\bar{l}}_t(x,z)\right)\left(\phi^{\bar{l}}(x+\zeta^k_t(x,z))-\phi^{\bar{l}}(x)\right) - \zeta^{k;i}_t(x,z)\partial_i\phi^l(x)\right)\pi^k_t(dz) \\ \mathcal{N}^{l\varrho}_t\phi(x) &:= \sigma^{1;i\varrho}_t(x)\partial_i\phi^l(x) + \upsilon^{1;l\bar{l}\varrho}_t(x)\phi^{\bar{l}}(x),\ \varrho\in\mathbf{N}, \\ \mathcal{I}^l_{t,z}\phi(x) &:= \left(\delta_{l\bar{l}}+\rho^{1;l\bar{l}}_t(x,z)\right)\phi^{\bar{l}}(x+\zeta^1_t(x,z)) - \phi^l(x). \end{aligned}$$

We assume that

$$\sigma^k_t(x) = (\sigma^{k;i\varrho}_{\omega,t}(x))_{1\le i\le d_1,\,\varrho\in\mathbf{N}}, \quad b_t(x) = (b^i_{\omega,t}(x))_{1\le i\le d_1}, \quad c_t(x) = (c^{l\bar{l}}_{\omega,t}(x))_{1\le l,\bar{l}\le d_2},$$

$$\upsilon^k_t(x) = (\upsilon^{k;l\bar{l}\varrho}_{\omega,t}(x))_{1\le l,\bar{l}\le d_2,\,\varrho\in\mathbf{N}}, \quad f_t(x) = (f^i_{\omega,t}(x))_{1\le i\le d_2}, \quad g_t(x) = (g^{i\varrho}_{\omega,t}(x))_{1\le i\le d_2,\,\varrho\in\mathbf{N}}$$

are random fields on $\Omega\times[0,T]\times\mathbf{R}^{d_1}$ that are $\mathcal{R}_T\otimes\mathcal{B}(\mathbf{R}^{d_1})$-measurable. Moreover, we assume that

$$\zeta^1_t(x,z) = (\zeta^{1;i}_{\omega,t}(x,z))_{1\le i\le d_1},\ \rho^1_t(x,z) = (\rho^{1;l\bar{l}}_{\omega,t}(x,z))_{1\le l,\bar{l}\le d_2},\ h^i_{\omega,t}(x,z))_{1\le i\le d_2},$$

are random fields on $\Omega\times[0,T]\times\mathbf{R}^{d_1}\times Z^1$ that are $\mathcal{P}_T\otimes\mathcal{B}(\mathbf{R}^{d_1})\otimes\mathcal{Z}^1$-measurable and

$$\zeta^2_t(x,z) = (\zeta^{2;i}_{\omega,t}(x,z))_{1\le i\le d_1},\ \rho^2_t(x,z) = (\rho^{2;l\bar{l}}_{\omega,t}(x,z))_{1\le l,\bar{l}\le d_2},$$

are random fields on $\Omega\times[0,T]\times\mathbf{R}^{d_1}\times Z^2$ that are $\mathcal{R}_T\otimes\mathcal{B}(\mathbf{R}^{d_1})\otimes\mathcal{Z}^2$-measurable. We also assume that there is a constant C such that $V_t\le C$ for all $(\omega,t)\in\Omega\times[0,T]$.

Let us introduce the following assumption for an integer $m\ge 1$ and a real number $\alpha^1\in(1,2)$. We remind the reader that the norms and seminorms $|\cdot|_0=[\cdot]_0$ and $|\cdot|_\beta$,

$\beta \in (0,1]$, were defined in the beginning of Section 2.2.

Let us introduce the following assumption for $m \in \mathbf{N}$ and a real number $\beta \in [0,2]$.

Assumption 4.3.1 (m, d_2)**.** *Let N_0 be a positive constant.*

(1) For all $(\omega, t, x) \in \Omega \times [0,T] \times \mathbf{R}^{d_1}$, the derivatives in x of the random fields b_t, c_t, σ_t^2, and υ_t^2 up to order m and σ_t^k and υ_t^k, $k \in \{1,2\}$, up to order $m+1$ exist, and for all $x \in \mathbf{R}^{d_1}$,

$$\max_{|\gamma|\le m}\Big(|\partial^\gamma b_t(x)| + |\partial^\gamma c_t(x)| + |\partial^\gamma \nabla\sigma_t^1(x)| + |\partial^\gamma \sigma_t^2(x)| + |\partial^\gamma\nabla\upsilon_t^1(x)| + |\partial^\gamma \upsilon_t^2(x)|\Big) \le N_0.$$

(2) For each $k \in \{1,2\}$ and all $(\omega, t, x, z) \in \Omega \times [0,T] \times \mathbf{R}^{d_1} \times Z^k$, the derivatives in x of the random field $\zeta_t^k(z)$ up to order m exist, and for all $x \in \mathbf{R}^{d_1}$,

$$\max_{|\gamma|\le m} |\partial^\gamma \zeta_t^1\,(x,z)\,| + \max_{|\gamma|=m}\Big[\partial^\gamma\zeta_t^1\,(\cdot,z)\Big]_{\frac{\beta}{2}} \le K_t^1\,(z)\,, \quad \forall z \in Z^1,$$

$$\max_{|\gamma|\le m} |\partial^\gamma \zeta_t^2\,(x,z)\,| \le K_t^2(z), \quad \forall z \in Z^2,$$

where K_t^1 (resp. K_t^2) are $\mathcal{P}_T \otimes \mathcal{Z}^1$ (resp. $\mathcal{P}_T \otimes \mathcal{Z}^1$) -measurable processes satisfying

$$\sup_{z\in Z^k} K_t^k(z) + \int_{Z^1} K_t^1(z)^\beta \pi_t^1(dz) + \int_{Z^2} K_t^2(z)^2\pi^2(dz) \le N_0.$$

(3) There is a constant $\eta < 1$ such that for each $k \in \{1,2\}$ on the set all $(\omega, t, x, z) \in \Omega \times [0,T] \times \mathbf{R}^{d_1} \times Z^k$ for which $|\nabla\zeta_t^k(x,z)| > \eta$,

$$\left|\Big(I_{d_1} + \nabla\zeta_t^k(x,z)\Big)^{-1}\right| \le N_0.$$

(4) For each $k \in \{1,2\}$ and all $(\omega, t, x, z) \in \Omega \times [0,T] \times \mathbf{R}^{d_1} \times Z^k$, the derivatives in x of the random field $\rho_t^k(z)$ up to order m exist, and for all $x \in \mathbf{R}^{d_1}$,

$$\max_{|\gamma|\le m} |\partial^\gamma \rho_t^1(x,z)| + \max_{|\gamma|=m}\Big[\rho_t^1\,(\cdot,z)\Big]_{\frac{\beta}{2}} \le l_t^1(z),$$

$$\max_{|\gamma|\le m} |\partial^\gamma \rho_t^2(x,z)| \le l_t^2\,(z)\,,$$

where l^1 (resp. l^2) is $\mathcal{P}_T \otimes \mathcal{Z}^1$ (resp. $\mathcal{P}_T \otimes \mathcal{Z}^2$) -measurable function satisfying

$$\int_{Z^1} l_t^1(z)^2\pi_t^1(dz) + \int_{Z^2} l_t^2(z)^2\pi^2(dz) \le N_0.$$

Let $L^2 = L^2(\mathbf{R}^{d_1}, \mathcal{B}(\mathbf{R}^{d_1}), \nu; \mathbf{R})$, where ν (differential is denoted by dx) is the Lebesgue measure. Let $\mathcal{S} = \mathcal{S}(\mathbf{R}^{d_1})$ be the Schwartz space of rapidly decreasing functions on $\mathbf{R}^{d_1}$.

The Fourier transform of an element $v \in \mathcal{S}$ is defined by

$$\hat{v}(\xi) = \mathcal{F}v(\xi) = \int_{\mathbf{R}^{d_1}} v(x)e^{-i2\pi\xi\cdot x}dx, \quad \xi \in \mathbf{R}^{d_1}.$$

We denote by $\mathcal{F}^{-1}$ its inverse. Denote the space of tempered distributions by $\mathcal{S}'$, the dual of $\mathcal{S}$.

Let $\Delta := \sum_{i=1}^{d_1} \partial_i^2$ be the Laplace operator on $\mathbf{R}^{d_1}$. For $\alpha \in \mathbf{R}$, we define the Sobolev scale

$$\begin{aligned} &H^\alpha(\mathbf{R}^{d_1};\mathbf{R}^d) \\ &= \left\{ v = (v^i)_{1\le i\le d} : v^i \in \mathcal{S}' \;\; and \;\; \left(1 + 4\pi^2 |\xi|^2\right)^{\alpha/2} \hat{v}^i \in L^2(\mathbf{R}^{d_1}), \quad \forall i \in \{1,\dots,d\} \right\} \\ &= \left\{ v = (v^i)_{1\le i\le d} : v^i \in \mathcal{S}' \;\; and \;\; (I-\Delta)^{\frac{\alpha}{2}} v^i \in L^2(\mathbf{R}^{d_1}), \quad \forall i \in \{1,\dots,d\} \right\} \end{aligned}$$

with the norm and inner product given by

$$\|v\|_{\alpha,d} = \left(\sum_{i=1}^d \left| \left(1 + 4\pi^2 |\xi|^2\right)^{\alpha/2} \hat{v}^i \right|_{L^2}^2 \right)^{1/2} = \left(\sum_{i=1}^d \left| (I-\Delta)^{\alpha/2} v^i \right|_{L^2}^2 \right)^{1/2}$$

and

$$(v,u)_{\alpha,d} = \sum_{i=1}^d \left((I-\Delta)^{\alpha/2} v^i, (I-\Delta)^{\alpha/2} u^i \right)_{L^2}, \quad \forall u, v \in H^\alpha(\mathbf{R}^{d_1};\mathbf{R}^d),$$

where

$$(I-\Delta)^{\alpha/2} v^i = \mathcal{F}^{-1}\left(\left(1 + 4\pi^2 |\xi|^2\right)^{\alpha/2} \hat{v}^i \right).$$

It is well-known that $C_c^\infty(\mathbf{R}^{d_1};\mathbf{R}^d)$ is dense in $H^\alpha(\mathbf{R}^{d_1};\mathbf{R}^d)$ for all $\alpha \in \mathbf{R}$. For $v \in H^1(\mathbf{R}^{d_1};\mathbf{R}^d)$ and $u \in H^{-1}(\mathbf{R}^{d_1};\mathbf{R}^d)$, we let

$$\langle v, u \rangle_{1,d} = (\Lambda^1 v, \Lambda^{-1} u)_{0,d},$$

and identify the dual of $H^1(\mathbf{R}^{d_1};\mathbf{R}^d)$ with $H^{-1}(\mathbf{R}^{d_1};\mathbf{R}^d)$ through this bilinear form. Moreover, all of the properties imposed in Section 4.2 for the abstract family of spaces $(H^\alpha)_{\alpha\in\mathbf{R}}$ and operators $(\Lambda^\alpha)_{\alpha\in\mathbf{R}}$ hold for the Sobolev scale. We refer the reader to [Tri10] for more details about the Sobolev scale (see the references therein as well).

For each $\alpha \in \mathbf{R}$, let $\mathbf{H}^\alpha(\mathbf{R}^{d_1};\mathbf{R}^d;\mathcal{F}_0)$ be the space of all $\mathcal{F}_0$-measurable $H^\alpha(\mathbf{R}^{d_1};\mathbf{R}^d)$-valued random variables $\tilde{\varphi}$ satisfying $\mathbf{E}\left[\|\tilde{\varphi}\|_\alpha^2\right] < \infty$.

Let $\mathbf{H}^\alpha(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$ be the space of all $H^\alpha(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$-valued $\mathcal{R}_T$-measurable processes f :

$\Omega\times[0,T]\to H^\alpha(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$ such that

$$\mathbf{E}\int_0^T \|f_t\|_\alpha^2 dV_t<\infty.$$

Let $\mathbf{H}^\alpha(\mathbf{R}^{d_1};\ell_2(\mathbf{R}^d))$ be the space of all sequences of $H^\alpha(\mathbf{R}^{d_1};\mathbf{R}^d)$-valued $\mathcal{P}_T$-measurable processes $\tilde g=(\tilde g^\varrho)_{\varrho\in\mathbf{N}}$, $\tilde g^\varrho:\Omega\times[0,T]\to H^\alpha(\mathbf{R}^{d_1};\mathbf{R}^d)$, satisfying

$$\mathbf{E}\int_0^T \|\tilde g_t\|_\alpha^2 dV_t=\mathbf{E}\int_0^T\sum_{\rho\in\mathbf{N}}\|\tilde g_t^\varrho\|_\alpha^2 dV_t<\infty.$$

Let $\mathbf{H}^\alpha(\mathbf{R}^{d_1};\mathbf{R}^d;\pi^1)$ be the space of all $H^\alpha(\mathbf{R}^{d_1};\mathbf{R}^d)$-valued $\mathcal{P}_T\otimes\mathcal{Z}^1$-measurable processes $\tilde h:\Omega\times[0,T]\times\mathcal{Z}^1\to H^\alpha(\mathbf{R}^{d_1};\mathbf{R}^d)$ such that

$$\mathbf{E}\int_0^T\int_{Z^1}\|\tilde h_t\,(z)\,\|_\alpha^2\pi_t^1(dz)dV_t<\infty..$$

For each $\alpha\in\mathbf{R}$, we set $H^\alpha=H^\alpha(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$, $\mathbf{H}^\alpha(\mathcal{F}_0)$, $\mathbf{H}^\alpha=\mathbf{H}^\alpha(\mathbf{R}^{d_1};\mathbf{R}^{d_2})$, $\mathbf{H}^\alpha(\ell_2)=\mathbf{H}^\alpha(\mathbf{R}^{d_1};\ell_2(\mathbf{R}^{d_2}))$, $\mathbf{H}^\alpha(\pi^1)=\mathbf{H}^\alpha(\mathbf{R}^{d_1};\mathbf{R}^{d_2};\pi^1)$, and $\|\cdot\|_\alpha=\|\cdot\|_{\alpha,d_2}$, $(\cdot,\cdot)_\alpha=(\cdot,\cdot)_{\alpha,d_2}$, $\langle\cdot,\cdot\rangle_1=\langle\cdot,\cdot\rangle_{1,d_2}$. We also set $C_c^\infty=C_c^\infty(\mathbf{R}^{d_1};\mathbf{R}^d)$.

Definition 4.3.1. Let $\varphi\in\mathbf{H}^0(\mathcal{F}_0)$, $f\in\mathbf{H}^{-1}, g\in\mathbf{H}^0(\ell_2)$, and $h\in\mathbf{H}^0(\pi^1)$. An H^0-valued strongly càdlàg process $u=(u_t)_{t\le T}$ is said to be a solution of the SIDE (4.3.1) if $u\in L^2(\Omega\times[0,T],\mathcal{O}_T,dV_td\mathbf{P};H^1)$ and $\mathbf{P}$-a.s. for all $t\in[0,T]$,

$$\begin{aligned}u_t\overset{H^{-1}}{=}\varphi&+\int_0^t\left((\mathcal{L}_s^{1;l}+\mathcal{L}_s^{2;l})u_s+b_s^i\partial_iu_s^l+c_s^{l\bar l}u_s^{\bar l}+f_s^l\right)dV_s+\int_0^t\left(\mathcal{N}_s^{l\varrho}u_s+g_s^{l\varrho}\right)dw_s^\varrho\\&+\int_0^t\int_{Z^1}\left(\mathcal{I}_{s,z}^lu_{s-}^{\bar l}+h_s^l(z)\right)\tilde\eta(ds,dz),\end{aligned}$$

where $\overset{H^{-1}}{=}$ indicates that the equality holds in the H^{-1}. That is, $\mathbf{P}$-a.s. for all $t\in[0,T]$ and $v\in H^1$,

$$\begin{aligned}(v,u_t)_0=(v,u_0)&+\int_0^t\langle v,(\mathcal{L}_s^1+\mathcal{L}_s^2)u_s+b_s^i\partial_iu_s+c_su_s+f_s\rangle_1dV_s\\&+\int_0^t\left(v,\left(\mathcal{N}_s^{l\varrho}u_s+g_s^{l\varrho}\right)\right)_0dw_s^\varrho+\int_0^t\int_{Z^1}\left(v,\left(\mathcal{I}_{s,z}^lu_{s-}^{\bar l}+h_s^l(z)\right)\right)\tilde\eta(ds,dz).\end{aligned}$$

The coming theorem is our existence result for equation (4.3.1). In the next section, we prove this theorem by appealing to Theorem 4.2.4.

Theorem 4.3.2. *Let Assumption 4.3.1(m,d_2) hold for $m\in\mathbf{N}$ and a real number $\beta\in[0,2]$. Then for every $\varphi\in\mathbf{H}^m(\mathcal{F}_0)$, $f\in\mathbf{H}^m$, $g\in\mathbf{H}^{m+1}(\ell_2), h\in\mathbf{H}^{m+\frac{\beta}{2}}(\pi^1)$, and there exists a unique solution $u=(u_t)_{t\le T}$ of (4.3.1) that is weakly càdlàg as an H^m-valued process and* *strongly càdlàg as an $H^{\alpha'}$-valued process for any $\alpha' < m$. Moreover, there is a constant $N = N(d_1, d_2, N_0, m, \eta, \beta)$ such that*

$$\mathbf{E}\left[\sup_{t\le T}\|u_t\|_m^2\right] \le N\mathbf{E}\left[\|\varphi\|_m^2 + \int_0^T\left(\|f_t\|_m^2 + \|g_t\|_{m+1}^2 + \int_{Z^1}\|h_t(z)\|^2_{m+\frac{\beta}{2}}\pi_t^1(dz)\right)dV_t\right].$$

The following corollary can be proved in the same way as Corollary 1 (with $p = 2$) in Chapter 4, Section 2.2 of [Roz90] by making use of the Sobolev embedding theorem.

Corollary 4.3.3. *Let Assumption 4.3.1(m, d_2) hold for an integer $m > \frac{d_1}{2}$ and a real number $\beta \in [0, 2]$. Then the solution $(u_t)_{t\le T}$ of (4.3.1) has a version with the following properties:*

(1) for every $x \in \mathbf{R}^{d_1}$, $u_t(x)$ is a $\mathcal{O}_T$-measurable $\mathbf{R}^{d_2}$-valued process;
(2) for $\beta = (2m - d_1)/2$ and for all $\omega \in \Omega$, $u \in D([0, T]; C^{\beta}_{loc}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))$;
(3) it possess all the properties mentioned in Theorem 4.3.2;
(4) for each bounded subset $Q \subseteq \mathbf{R}^d$,

$$\mathbf{E}\left[\sup_{t\le T}|u_t|^2_{\beta;Q;\mathbf{R}^{d_2}}\right] + \mathbf{E}\int_{]0,T]}\|u_t\|^2_{\beta;Q;\mathbf{R}^{d_2}}dV_t$$
$$\le N\mathbf{E}\left[\|\varphi\|_m^2 + \int_0^T\left(\|f_t\|_m^2 + \|g_t\|_{m+1}^2 + \int_{Z^1}\|h_t(z)\|^2_{m+\frac{\beta}{2}}\pi_t^1(dz)\right)dV_t\right],$$

where $N = N(d_1, d_2, N_0, m, \eta, \beta)$ is a constant;
(5) if $(u_t^1)_{t\le T}$ and $(u_t^2)_{t\le T}$ are solutions of (4.3.1) possessing properties (1) and (2), then

$$\mathbf{P}\left(\sup_{t\le T, x\in\mathbf{R}^{d_1}}|u_t^1(x) - u_t^2(x)| > 0\right) = 0.$$

Remark 4.3.4. If $m > \alpha_1 \vee \alpha_2 + \frac{d}{2}$, then $u \in D([0, T]; C^{\alpha_1\vee\alpha_2}_{loc}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))$ and we can obtain a representation of the solution of (4.3.1) as in Theorem 3.2.2 by applying the Ito-Wentzell formula given in Proposition 3.4.16. However, there are two important points to notice. First, since we have only established an integer regularity theory for our equation, then the best we can hope for is a theory for classical solutions with integer assumptions on the coefficients, initial condition, and free terms. This is in stark contrast with Chapter 3. Moreover, the restriction to $p = 2$ has dire consequences, since the $\frac{d_1}{2}$ term gets larger as d_1 grows, and hence one must impose more regularity as the dimension grows. This is a strong motivation to consider the L^p-Sobolev theory for (4.3.1) in weighted spaces, so that the $\frac{d_1}{2}$ can be replaced with $\frac{d_1}{p}$. This way, by taking p large, the term $\frac{d_1}{p}$ can be made as small as one likes. It is worth mentioning that we could have considered weighted scale of Sobolev spaces here (see, e.g. [GK92]), but we will leave this for a future project. At the time of writing, the integer scale L^p-Sobolev theory for (degenerate) (4.3.1) is currently underway and will be available soon.

4.3.2 Proof of Theorem 4.3.2

By [MR99] (see Examples 2.3-2.4), the stochastic integrals in (4.3.1) can be written as stochastic integrals with respect to a cylindrical martingale. We will apply Theorem 4.2.4 to (4.3.1) with $\alpha = m$ and $\mu = 1$ by checking that Assumptions 4.2.1(λ, 1) and 4.2.2(λ, 1) for $\lambda \in \{0, m\}$ are implied by Assumption 4.3.1(m, d_2). We start with $\lambda = 0$ as our base case and show that $\lambda = m$ can be reduced to it.

We introduce our base assumption for $\beta \in [0, 2]$.

Assumption 4.3.2 (d_2)**.** *Let N_0 be a positive constant.*

(1) For all $(\omega, t, x) \in \Omega \times [0, T] \times \mathbf{R}^{d_1}$, the derivatives in x of the random fields $b_t, \sigma^1_t, \sigma^2_t$, and $\operatorname{div} \sigma^1_t$ exist, and for all $x \in \mathbf{R}^{d_1}$,

$$|\nabla \operatorname{div} \sigma^1_t (x)| + |\sigma^k_t (x)| + |\nabla \sigma^k_t (x)| + |\operatorname{div} b_t(x)| + |c_t(x)| + |\upsilon^2_{t,\mathrm{sym}} (x)| + |\nabla \upsilon^1_t (x)| \le N_0.$$

(2) For each $k \in \{1, 2\}$ and all $(\omega, t, x, z) \in \Omega \times [0, T] \times \mathbf{R}^{d_1} \times Z^k$, the derivatives in x of the random fields $\zeta^k_t(z)$ exist, and for all $x \in \mathbf{R}^{d_1}$,

$$|\zeta^1_t(x, z)| \le K^1_t (z), \quad |\nabla \zeta^1_t (x, z)| \le \bar{K}^1_t (z), \quad \left[\operatorname{div} \zeta^1_t (\cdot, z)\right]_{\frac{\beta}{2}} \le \tilde{K}^1_t (z), \quad \forall z \in Z^1,$$
$$|\zeta^2_t(x, z)| \le K^2_t (z), \quad |\nabla \zeta^2_t (x, z)| \le \bar{K}^2_t (z), \quad \forall z \in Z^2,$$

where $K^1_t, \bar{K}^1_t, \tilde{K}^1_t$ (resp. $K^2_t, \bar{K}^2_t$) are $\mathcal{P}_T \otimes \mathcal{Z}^1$ (resp. $\mathcal{P}_T \otimes \mathcal{Z}^2$) -measurable processes satisfying

$$\sup_{z \in Z^1} \left(K^1_t(z) + \bar{K}^1_t(z) + \tilde{K}^1_t (z)\right) + \int_{Z^1} \left(K^1_t(z)^\beta + \bar{K}^1_t (z)^2 + \tilde{K}^1_t (z)^2\right) \pi^1_t(dz) \le N_0,$$
$$\sup_{z \in Z^2} \left(K^2_t(z) + \bar{K}^2_t(z)\right) + \int_{Z^2} \bar{K}^2_t(z)^2 \pi^2(dz) \le N_0.$$

(3) There is a constant $\eta < 1$ such that for each $k \in \{1, 2\}$ on the set all $(\omega, t, x, z) \in \Omega \times [0, T] \times \mathbf{R}^{d_1} \times Z^k$ for which $|\nabla \zeta^k_t(x, z)| > \eta$,

$$\left|\left(I_{d_1} + \nabla \zeta^k_t(x, z)\right)^{-1}\right| \le N_0.$$

(4) For each $k \in \{1, 2\}$ and all $(\omega, t, x, z) \in \Omega \times [0, T] \times \mathbf{R}^{d_1} \times Z^k$, $|\rho^k_t(x, z)| \le l^k_t(z)$, and for all $(\omega, t, z) \in \Omega \times [0, T] \times Z^1$, $\left[\rho^1_{t,\mathrm{sym}}(\cdot, z)\right]_{\frac{\beta}{2}} \le \tilde{l}^1_t (z)$, where l^k (resp., $\tilde{l}^1$) is a $\mathcal{P}_T \otimes \mathcal{Z}^k$-measurable (resp. $\mathcal{P}_T \otimes \mathcal{Z}^1$-measurable) functions satisfying

$$\int_{Z^1} \left(l^1_t(z)^2 + \tilde{l}^1_t (z)^2\right) \pi^1(dz) + \int_{Z^2} l^2_t(z)^2 \pi^2(dz) \le N_0.$$

Note that Assumption 4.3.2(d_2) is weaker than Assumption 4.3.1($0, d_2$).

Let us make the following convention for the remainder of this section. If we do not specify to which space the parameters ω, t, x, y, and z belong, then we mean $\omega \in \Omega$, $t \in [0,T]$, $x \in \mathbf{R}^{d_1}$, and $z \in Z^k$. Moreover, unless otherwise specified, all statements hold for all ω, t, x, y, and z independent of any constant N introduced is independent of ω, t, x, y, and z. We will also drop the dependence of processes t, x, and z when we feel it will not obscure our argument. Lastly, all derivatives and Hölder norms are taken with respect to $x \in \mathbf{R}^{d_1}$.

Remark 4.3.5. Let Assumption 4.3.2(d_2) hold. For each k and $\theta \in [0,1]$, on the set all ω, t, and z in which $|K_t^k(z)| \le \eta$, we have

$$\left|(I_{d_1} + \theta \nabla \zeta_t^k(x,z))^{-1}\right| \le \frac{1}{1-\theta\eta}.$$

Moreover, for each k and all ω, t, and z, we have

$$\left|(I_{d_1} + \nabla \zeta_t^k(x,z))^{-1}\right| \le \max\left(\frac{1}{1-\theta\eta}, N_0\right).$$

Therefore, by Hadamard's theorem (see, e.g., Theorem 0.2 in [DMGZ94] or 51.5 in [Ber77]):

- for each k and $\theta \in [0,1]$, on the set all ω, t, and z in which $|K_t^k(z)| \le \eta$, the mapping

  $$\tilde{\zeta}_{t,\theta}^k(x,z) := x + \theta \zeta_t^k(x,z)$$

  is a global diffeomorphism in x;

- for each k and all ω, t, and z, the mapping

  $$\tilde{\zeta}_t^k(x,z) = \tilde{\zeta}_{t,1}^k(x,z) = x + \zeta_t^k(x,z)$$

  is a global diffeomorphism in x.

When inverse of the mapping $x \mapsto \tilde{\zeta}_{t,\theta}^k(x,z)$ exists, we denote it by

$$\tilde{\zeta}_{t,\theta}^{k;-1}(x,z) = \left(\tilde{\zeta}_{t,\theta}^{k;-1;j}(x,z)\right)_{1\le j\le d_1}$$

and note that

$$\tilde{\zeta}_{t,\theta}^{k;-1}(x,z) = x - \theta \zeta_t^k(\tilde{\zeta}_{t,\theta}^{k;-1}(x,z), z).$$

Furthermore, for each k and $\theta \in [0,1]$, on the set all ω, t, and z in which $|K_t^k(z)| \le \eta$, there is a constant $N = N(d_1, N_0, \eta)$ such that

$$|\nabla \tilde{\zeta}^{k;-1}_{t,\theta}(x,z)| \leq N \tag{4.3.2}$$

and for each k and all ω, t, and z,

$$|\nabla \tilde{\zeta}^{k;-1}_{t}(x,z)| \leq N. \tag{4.3.3}$$

Using simple properties of the determinant, we can easily show that there is a constant $N = N(d_1)$ such that for an arbitrary real-valued $d_1 \times d_1$ matrix A,

$$|\det(I_{d_1} + A) - 1| \leq N|A| \quad \textit{and} \quad |\det(I_{d_1} + A) - 1 - \operatorname{tr} A| \leq N|A|^2.$$

Thus, there is a constant $N = N(d_1, N_0, \eta)$ such that

$$|\det \nabla \tilde{\zeta}^{k;-1} - 1| = \left|\det\left(I_d - \zeta^k_t(\tilde{\zeta}^{k;-1}_t)\right) - 1\right| \leq N|\nabla \zeta^k(\tilde{\zeta}^{k;-1})| \tag{4.3.4}$$

and

$$\left|\det \nabla \tilde{\zeta}^{k;-1} - 1 + \operatorname{div}\left(\zeta^k(\tilde{\zeta}^{k;-1})\right)\right| \leq N|\nabla \zeta^k(\tilde{\zeta}^{k;-1})|^2.$$

Since $\partial_l \tilde{\zeta}^{k;-1;j} = \delta_{lj} - \partial_m \zeta^{k;j}(\tilde{\zeta}^{k;-1})\partial_l \tilde{\zeta}^{k;-1;m})$, we have

$$|\operatorname{div}(\zeta^k(\tilde{\zeta}^{k;-1})) - \operatorname{div} \zeta^k(\tilde{\zeta}^{k;-1})| = |\partial_j \zeta^{k;l}(\tilde{\zeta}^{k;-1}_t)(\partial_l \tilde{\zeta}^{k;-1;j} - \delta_{lj})| \leq N|\nabla \zeta^k(\tilde{\zeta}^{k;-1})|^2,$$

and thus

$$\left|\det \nabla \tilde{\zeta}^{k;-1} - 1 + \operatorname{div} \zeta^k(\tilde{\zeta}^{k;-1})\right| \leq N|\nabla \zeta^k(\tilde{\zeta}^{k;-1})|^2. \tag{4.3.5}$$

In the following three lemmas, we will show that Assumptions 4.2.1(λ, 1) and 4.2.2(λ, 1) for $\lambda \in \{0, m\}$ hold under Assumption 4.3.2(β) for any $\beta \in [0, 2]$. For each $l \in \{1, \ldots, d_2\}$ and all $\phi \in C^\infty_c$, let

$$\mathcal{L}^l_t \phi = (\mathcal{L}^{1;l}_t + \mathcal{L}^{2;l}_t)\phi + b^i_t \partial_i \phi^l + c^{l\bar{l}}_t \phi^{\bar{l}},$$
$$\mathcal{A}^{1;l}_t \phi = \frac{1}{2}\sigma^{1;i\varrho}_t \sigma^{1;j\varrho}_t \partial_{ij}\phi^l + \sigma^{1;i\varrho}_t \upsilon^{1;l\bar{l}\varrho}_t \partial_i \phi^{\bar{l}}, \quad \textit{and} \quad \mathcal{J}^1_t \phi = \mathcal{L}^1_t \phi - \mathcal{A}^1_t \phi.$$

Lemma 4.3.6. *Let Assumption 4.3.2(d_2) hold. Then there is a constant $N = N(d_1, d_2, N_0, \eta)$ such that for all $(\omega, t) \in \Omega \times [0, T]$ and $v \in H^1$,*

$$\|\mathcal{L}_t v\|_{-1} \leq N\|v\|_1, \quad \|\mathcal{A}_t v\|_{-1} \leq N\|v\|_1, \quad \|\mathcal{J}^1_t v\|_{-1} \leq N\|v\|_1,$$
$$\|\mathcal{N}_t v\|_0 \leq N\|\phi\|_1, \quad \text{and} \quad \int_{Z^1} \|\mathcal{I}_{t,z} v\|_0^2 \pi^1_t(dz) \leq N\|v\|_1^2.$$

Proof. First we will show that there is a constant N such that

$$(\psi, \mathcal{L}_t\phi)_0 \le N\|\psi\|_1\|\phi\|_1, \quad \forall \phi \in C_c^\infty.$$

Once this is established, we know that $\mathcal{L}$ extends to a linear operator from H^1 to H^{-1} (still denoted by $\mathcal{L}$) and $\|\mathcal{L}_t v\|_{-1} \le N\|v\|_1$, for all $v \in H^1$. Using Taylor's formula and the divergence theorem, we get that for all and all $\phi, \psi \in C_c^\infty$

$$(\psi, \mathcal{L}\phi)_0 = \sum_{k=1}^{2} \left((\psi, \mathfrak{L}_t^k \phi)_0 + (\partial_i \psi, \mathfrak{Y}_t^{k;i}\phi)_0 + (\psi, b_t \partial_i \phi)_0 + (\psi, c_t\phi)_0 \right),$$

where for each $k \in \{1,2\}$, $l \in \{1,\dots,d_2\}$, and $i \in \{1,2,\dots,d_1\}$,

$$\begin{aligned}
\mathfrak{L}^{k;l}\phi := &- \int_{\bar{K}^k<\eta} \int_0^1 \left(\phi^l(\tilde{\zeta}^k_\theta) - \phi^l\right) \partial_i \zeta^{k;i} d\theta \pi^k(dz) \\
&- \int_{\bar{K}^k<\eta} \int_0^1 \theta \partial_j \phi^l(\tilde{\zeta}^k_\theta) \partial_i \zeta^{k;j} \zeta^{k;i} d\theta \pi^k(dz) \\
&+ \int_{Z^k} \rho^{k;l\bar{l}} \left(\phi^l(\tilde{\zeta}^k) - \phi^l\right) \pi^k(dz) + \sigma^{k;i\varrho} \upsilon^{k;l\bar{l}\varrho} \partial_i \phi^{\bar{l}} \\
&+ \int_{\bar{K}^k>\eta} \left(\phi^l(\tilde{\zeta}^k) - \phi^l - \zeta^{k;i}\partial_i \phi^l\right) \pi^k(dz), \\
\mathfrak{Y}^{k;li}\phi := &- \int_{\bar{K}^k<\eta} \int_0^1 \left(\phi^l(\tilde{\zeta}^k_\theta) - \phi^l\right) \zeta^{k;i} d\theta \pi^k(dz) - \frac{1}{2}\partial_i \left(\sigma^{k;i\varrho}\sigma^{k;j\varrho}\right) \partial_j \phi^l.
\end{aligned}$$

For the remainder of the proof, we make the convention that statements hold for all $\phi, \psi \in C_c^\infty$ and that all constants N are independent of ϕ. By Minkowski's integral inequality and Hölder's inequality, we have (using the notation of Remark 4.3.5)

$$\begin{aligned}
\|\mathfrak{L}^k\phi\|_0 \le &\left(\int_{\bar{K}^k<\eta} (K^k)^2 \pi^k(dz) \right)^{\frac{1}{2}} \int_0^1 \left(\int_{\mathbf{R}^{d_1}} \int_{\bar{K}^k<\eta} |\phi(\tilde{\zeta}^k_\theta(z)) - \phi|^2 \pi^k(dz) dx \right)^{\frac{1}{2}} d\theta \\
&+ \int_{\bar{K}^k<\eta} K^k(z)^2 \int_0^1 \left(\int_{\mathbf{R}^{d_1}} |\nabla \phi(\tilde{\zeta}^k_\theta)|^2 dx \right)^{\frac{1}{2}} \pi^k(dz) \theta d\theta \\
&+ \left(\int_{Z^1} (l^k)^2 \pi^k(dz) \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^{d_1}} \int_{Z^1} |\phi(\tilde{\zeta}^k) - \phi|^2 \pi^k(dz) dx \right)^{\frac{1}{2}} \\
&+ N\|\nabla\phi\|_0 + \int_{\bar{K}^k \ge \eta} \int_0^1 \left(\int_{\mathbf{R}^{d_1}} |\phi(\tilde{\zeta}^k) - \phi - \zeta^{k;i}\partial_i \phi|^2 dx \right)^{\frac{1}{2}} d\theta \pi^k(dz),
\end{aligned}$$

and for all $i \in \{1,\dots,d_1\}$,

$$\|\mathfrak{Y}^{k;i}\phi\|_0 \le \left(\int_{\bar{K}^k<\eta} (K^k)^2 \pi^k(dz) \right)^{\frac{1}{2}} \int_0^1 \left(\int_{\mathbf{R}^{d_1}} \int_{\bar{K}^k<\eta} |\phi(\tilde{\zeta}^k_\theta) - \phi|^2 \pi^k(dz) dx \right)^{\frac{1}{2}} d\theta + N\|\nabla\phi\|_0.$$

Applying the change of variable formula and appealing to (4.3.2), we find that

$$\int_{\mathbf{R}^{d_1}}\int_{\bar{K}^k<\eta}|\phi(\tilde{\zeta}^k_\theta)-\phi|^2\pi^k(dz)dx\leq\theta\int_{\bar{K}^k<\eta}\int_0^1\int_{\mathbf{R}^{d_1}}|\nabla\phi(\tilde{\zeta}^k_{\theta\bar{\theta}})|^2|\zeta^k|^2dxd\bar{\theta}$$
$$\leq\theta\int_{\bar{K}^k<\eta}(K^k)^2\int_0^1\int_{\mathbf{R}^{d_1}}|\nabla\phi|^2\left|\det\nabla\tilde{\zeta}^{k;-1}_{\theta\bar{\theta}}\right|dxd\bar{\theta}\leq N\theta\|\nabla\phi\|_0^2.$$

Similarly, since $\pi^k(\{z\in Z^k:\bar{K}^k\geq\eta\})\leq N_0$, we have

$$\int_{\mathbf{R}^{d_1}}\int_{\bar{K}^k\geq\eta}|\phi(\tilde{\zeta}^k)-\phi|^2\pi^k(dz)dx\leq 2\int_{\mathbf{R}^{d_1}}\int_{\bar{K}^k\geq\eta}\left(|\phi(\tilde{\zeta}^k)|^2+|\phi|^2\right)\pi^k(dz)dx$$
$$\leq\int_{\bar{K}^k\geq\eta}\int_{\mathbf{R}^{d_1}}|\phi|^2\left(1+\left|\det\nabla\tilde{\zeta}^{k;-1}\right|\right)dx\pi^k(dz)\leq N\|\phi\|_0^2$$

and

$$\int_{\bar{K}^k\geq\eta}\left(\int_{\mathbf{R}^{d_1}}|\phi(\tilde{\zeta}^k)-\phi-\zeta^{k;i}\partial_i\phi|^2dx\right)^{\frac{1}{2}}\pi^k(dz)$$
$$\leq N\int_{\bar{K}^k\geq\eta}\left(\int_{\mathbf{R}^{d_1}}\left(|\phi|^2\left(1+\left|\det\nabla\tilde{\zeta}^{k;-1}\right|\right)+(K^k)^2|\nabla\phi|^2\right)dx\right)^{\frac{1}{2}}\pi^k(dz)\leq N\|\phi\|_1,$$

where in the last inequality we used (4.3.3). Moreover,

$$\int_{\bar{K}^k<\eta}(K^k)^2\left(\int_{\mathbf{R}^{d_1}}|\nabla\phi(\tilde{\zeta}^k_\theta)|^2dx\right)^{\frac{1}{2}}\pi^k(dz)$$
$$\leq\int_{\bar{K}^k<\eta}(K^k)^2\left(\int_{\mathbf{R}^{d_1}}|\nabla\phi|^2\det\nabla\tilde{\zeta}^{k;-1}_\theta\,dx\right)^{\frac{1}{2}}\pi^k(dz)\leq N\|\nabla\phi\|_0.$$

Combining the above estimates, we get that $(\psi,\mathcal{L}\phi)_0\leq N\|\psi\|_1\|\phi\|_1$. It is clear from the above computation that

$$\|\mathcal{A}^1_t\phi\|_{-1}\leq N\|\phi\|_1,\quad\|\mathcal{J}^1_t\phi\|_{-1}\leq N\|\phi\|_1,$$

where actually $\mathcal{A}^1$ and $\mathcal{J}^1$ are actually extensions of the operators defined above. The inequality $\|\mathcal{N}\phi\|_0\leq N\|\phi\|_1$ can easily be obtained. Following similar calculations to ones we derived above (using (4.3.3) and (4.3.2)), we obtain

$$\int_{Z^1}\|\mathcal{I}\phi\|_0^2\pi^1(dz)\leq N(A_1+A_2),$$

where

$$A_1 := \int_{\mathbf{R}^{d_1}} \int_{\bar{K}^1 \le \eta} \int_0^1 |\nabla \phi(\tilde{\zeta}^1_\theta)|^2 |\xi^1|^2 \pi^1(dz) d\theta dx + \int_{\mathbf{R}^{d_1}} \int_{K^1 > \eta} \left(|\phi(\tilde{\zeta}^1)|^2 + |\phi|^2 \right) \pi^1(dz) dx \le N \|\phi\|_1^2$$

and

$$A_2 := \int_{\mathbf{R}^{d_1}} \int_{Z^1} \rho^{1;l\bar{l}} \phi^{\bar{l}}(\tilde{\zeta}^1) \pi^1 (dz)\, dx \le N \|\phi\|_0^2.$$

□

Lemma 4.3.7. *Let Assumption 4.3.2(d_2) hold. Then there is a constant $N = N(d_1, d_2, N_0, \eta, \beta)$ such that for all $(\omega, t) \in \Omega \times [0, T]$ and all $v \in H^1$,*

$$2\langle v, \mathcal{L}^2_t v + b^i_t \partial_i v_t + c^{\bar{l}}_t v^{\bar{l}}_t \rangle_1 + \frac{1}{4} (\sigma^{2;i\varrho}_t \partial_i v, \sigma^{2;j\varrho}_t \partial_j v)_0 + \frac{1}{4} \int_{Z^2} \|v(\tilde{\zeta}^2_t(z)) - v\|_0^2 \pi^2_t(dz) \le N \|v\|_0^2, \tag{4.3.6}$$

$$2\langle v, \mathcal{A}^1_t v \rangle_1 + \|\mathcal{N}_t v\|_0^2 \le N \|v\|_0^2, \quad 2\langle v, \mathcal{J}^1_t v \rangle_1 + \int_{Z^1} \|\mathcal{I}_{t,z} v\|_0^2 \pi^1_t(dz) \le N \|v\|_0^2, \tag{4.3.7}$$

and

$$2\langle v, \mathcal{L}_t v + f_t \rangle_1 + \|\mathcal{N}_t v + g_t\|_0^2 + \int_{Z_1} \|\mathcal{I}_{t,z} v + h_t(z)\|_0^2 \pi^1 (dz) + \frac{1}{4} (\sigma^{2;i\varrho}_t \partial_i v, \sigma^{2;j\varrho}_t \partial_j v)_0 + \frac{1}{4} \int_{Z^2} \|v(\tilde{\zeta}^2_t(z)) - v\|_0^2 \pi^2_t(dz) \le N \left(\|v\|_0^2 + \|f_t\|_0^2 + \|g_t\|_1^2 + \int_{Z^1} \|h_t(z)\|^2_{\frac{\beta}{2}} \pi^1_t(dz) \right).$$

Proof. For the remainder of the proof, we make the convention that statements hold for all $\phi \in C^\infty_c$ and that all constants N are independent of ϕ. Using the divergence theorem, we get

$$2\langle \phi, \mathcal{A}^1 \phi \rangle_1 + \|\mathcal{N}\phi\|_0^2 = \frac{1}{2} \int_{\mathbf{R}^{d_1}} \left(|\operatorname{div} \sigma^1|^2 + 2\sigma^{1;i} \partial_i \operatorname{div} \sigma^1 + \partial_j \sigma^{1;i} \partial_i \sigma^{1;j} \right) |\phi|^2 dx + \int_{\mathbf{R}^{d_1}} \left(|\upsilon^1 \phi|^2 - 2\phi^l \left(\upsilon^{1;l\bar{l}}_{\text{sym}} \operatorname{div} \sigma^1 + \sigma^{1;i} \partial_i \upsilon^{1;l\bar{l}}_{\text{sym}} \right) \phi^{\bar{l}} \right) dx \le N \|\phi\|_0^2.$$

Rearranging terms and using the identity $2a(b-a) = -|b-a|^2 + |b|^2 - |a|^2$, $a, b \in \mathbf{R}$, we obtain

$$2\langle \phi, \mathcal{J}^1 \phi \rangle_1 + \int_{Z_1} \|\mathcal{I}\phi\|_0^2 \pi^1_t(dz) = A_1 + A_2,$$

where

$$A_1 := \int_{\mathbf{R}^{d_1}}\int_{Z^1}\left(|\phi(\tilde{\zeta}^1)|^2 - |\phi|^2 - 2\phi\zeta^{1;i}\partial_i\phi\right)\pi^1(dz)dx$$

$$A_2 := 2\int_{\mathbf{R}^{d_1}}\int_{Z^1}\left(\phi^l(\tilde{\zeta}^1)\rho^{1;l\bar{l}}\phi^{\bar{l}}(\tilde{\zeta}^1) - \phi^l\rho^{1;\bar{l}}\phi^l\right)\pi^1(dz)dx + \int_{\mathbf{R}^{d_1}}\int_{Z^1}|\rho^1\phi(\tilde{\zeta}^1)|^2\pi^1(dz)dx,$$

Since

$$|\operatorname{div}\zeta^1(\tilde{\zeta}^{1;-1}) - \operatorname{div}\zeta^1| \le [\operatorname{div}\zeta^1]_{\frac{\beta}{2}}(K^1)^{\frac{\beta}{2}} \le (\bar{K}^1)^2 + (K^1)^\beta,$$

changing the variable of integration and making use of the estimate (4.3.5), we obtain

$$A_1 \le \int_{\mathbf{R}^{d_1}}|\phi|^2\int_{Z^1}|\det\nabla\tilde{\zeta}^{1;-1} - 1 + \operatorname{div}\zeta^1|\pi^1(dz)dx \le N\|\phi\|_0^2$$

and

$$\begin{aligned}A_2 &= 2\int_{\mathbf{R}^{d_1}}\int_{Z^1}\phi^l\left(\rho^{1;l\bar{l}}(\tilde{\zeta}^{1;-1}) - \rho^{1;l\bar{l}}\right)\phi^{\bar{l}}\pi^1(dz)dx\\ &\quad + \int_{\mathbf{R}^{d_1}}\int_{Z^1}2\phi^l\rho^{1;l\bar{l}}(\tilde{\zeta}^{1;-1})\phi^{\bar{l}}\left(\det\nabla\tilde{\zeta}^{1;-1} - 1\right)\pi^1(dz)dx\\ &\quad + \int_{\mathbf{R}^{d_1}}\int_{Z^1}|\rho^1(\tilde{\zeta}^{1;-1})\phi|^2\det\nabla\tilde{\zeta}^{1;-1}\pi^1(dz)dx =: A_{21} + A_{22} + A_{23}.\end{aligned}$$

Owing to (4.3.4) and Hölder's inequality, we have

$$A_{22} + A_{23} \le N\int_{Z^1}((l^1)^2 + (K^1)^2)\pi^1(dz)\|\phi\|_0^2.$$

For $\beta > 0$, we have

$$A_{21} \le N\int_{Z^1}\left[\rho^1_{\mathrm{sym}}\right]_{\frac{\beta}{2}}(K^1)^{\frac{\beta}{2}}\pi^1(dz)\|\phi\|_0^2 \le N\int_{Z^1}\left((\bar{l}^1)^2 + (K^1)^\beta\right)\pi^1(dz)\|\phi\|_0^2 \le N\|\phi\|_0^2$$

and for $\beta = 0$, using Holder's inequality, we get

$$A_{21} \le N\|\phi\|_0^2\int_{Z^1}(l^1)^2\pi^1(dz).$$

By the divergence theorem, we have

$$2\langle\phi, \mathcal{L}^2\phi\rangle_0 = B_1 + B_2 + B_3,$$

where

$$B_1 := \int_{\mathbf{R}^{d_1}}\left(\phi^l\sigma^{2;i\varrho}\sigma^{2;j\varrho}\partial_{ij}\phi^l + 2\sigma^{2;i\varrho}\upsilon^{2;l\bar{l}\varrho}\partial_i\phi^{\bar{l}}\right)dx,$$

$$
\begin{aligned}
B_2 &:= 2\int_{\mathbf{R}^{d_1}}\int_{Z^2}\phi^l\left(\phi^l(\tilde\zeta^2)-\phi^l-\zeta^{2;i}\partial_i\phi^l\right)\pi^2(dz)dx,\\
B_3 &:= 2\int_{\mathbf{R}^{d_1}}\int_{Z^2}\phi^l\rho^{2;l\bar l}\left(\phi^{\bar l}(\tilde\zeta^2)-\phi^{\bar l}\right)\pi^2(dz)dx.
\end{aligned}
$$

Owing to the divergence theorem, we have

$$
(\phi,\sigma^{2;i\varrho}\sigma^{2;j\varrho}\partial_{ij}\phi)_0=-\left(\left(\sigma^{2;i\varrho}\sigma^{2;j\varrho}\partial_i\phi+\phi\left(\sigma^{2;j\varrho}\operatorname{div}\sigma^{2;\varrho}+\sigma^{2;i\varrho}\partial_i\sigma^{2;j\varrho}\right)\right),\partial_j\phi\right)_0
$$

$$
\begin{aligned}
(\phi\sigma^{2;i\varrho}\partial_i\sigma^{2;j\varrho},\partial_j\phi)_0 &= -\frac{1}{2}\left(\phi\left(\partial_j\sigma^{2;i\varrho}\partial_i\sigma^{2;j\varrho}+\sigma^{2;j\varrho}\partial_j\operatorname{div}\sigma^{2;\varrho}\right),\phi\right)_0,\\
(\phi\sigma^{2;j\varrho}\partial_j\operatorname{div}\sigma^{2;\varrho},\phi)_0 &= -(\phi|\operatorname{div}\sigma^2|^2,\phi)_0+2(\phi\sigma^{2;i\varrho}\operatorname{div}\sigma^{2;\varrho},\partial_j\phi)_0,
\end{aligned}
$$

and hence,

$$
\begin{aligned}
(\phi,\sigma_t^{2;i\varrho}\sigma_t^{2;j\varrho}\partial_{ij}\phi)_0 = &-\left(\sigma^{2;i\varrho}\sigma^{2;j\varrho}\partial_i\phi^l\partial_j\phi^l+2\phi\sigma^{2;j\varrho}\operatorname{div}\sigma^{2;\varrho},\partial_j\phi\right)_0\\
&+\frac{1}{2}\left(\phi\left(\partial_j\sigma^{2;i\varrho}\partial_i\sigma^{2;j\varrho}-|\operatorname{div}\sigma^2|^2\right),\phi\right)_0.
\end{aligned}
$$

Thus, by Young's inequality,

$$
B_1\le-\frac{1}{2}\int\partial_i\phi^l\sigma^{2;i\varrho}\sigma^{2;j\varrho}\partial_j\phi^l dx+N\|\phi\|_0^2.
$$

Once again making use of the identity $2a(b-a)=-|b-a|^2+|b|^2-|a|^2$, $a,b\in\mathbf{R}$, we get

$$
2\phi^l(\phi^l(\tilde\zeta^2)-\phi^l-\zeta^{2;i}\partial_i\phi^l)=-|\phi(\tilde\zeta^2)-\phi|^2+|\phi(\tilde\zeta^2)|^2-|\phi|^2-\zeta_t^{2;i}\partial_i|\phi|^2.
$$

Changing the variable of integration and applying the divergence theorem, we obtain

$$
\begin{aligned}
B_2 = &-\int_{\mathbf{R}^{d_1}}\int_{Z^2}|\phi(\tilde\zeta^2)-\phi|^2\pi^2(dz)dx\\
&+\int_{\mathbf{R}^{d_1}}\int_{Z^2}|\phi|^2\left(\det\nabla\tilde\zeta^{2;-1}-1+\operatorname{div}\zeta^2(\tilde\zeta^{2;-1})\right)\pi^2(dz)dx\\
&+\int_{\mathbf{R}^{d_1}}\int_{Z^2}|\phi|^2\left(\operatorname{div}\zeta^2-\operatorname{div}\zeta^2(\tilde\zeta^{2;-1})\right)\pi^2(dz)dx.
\end{aligned}
$$

Changing the variable of integration in the last term of B_2, we get

$$
\begin{aligned}
&\int_{\mathbf{R}^{d_1}}\int_{Z^2}|\phi|^2\left(\operatorname{div}\zeta^2-\operatorname{div}\zeta^2(\tilde\zeta^{2;-1})\right)\pi^2(dz)dx\\
&=\int_{\mathbf{R}^{d_1}}\int_{Z^2}\left(|\phi|^2-|\phi(\tilde\zeta)|^2\det\nabla\tilde\zeta^2\right)\operatorname{div}\zeta^2\pi^2(dz)dx\\
&=\int_{\mathbf{R}^{d_1}}\int_{Z^2}|\phi(\tilde\zeta^2)|^2\left(1-\det\nabla\tilde\zeta^2\right)\operatorname{div}\zeta^2\pi^2(dz)dx
\end{aligned}
$$

$$+\int_{\mathbf{R}^{d_1}}\int_{Z^2}(\phi^l-\phi^l(\tilde{\zeta}^2))(\phi^l+\phi^l(\tilde{\zeta}^2))\operatorname{div}\zeta^2\pi^2(dz)dx =: B_{21}+B_{22}.$$

Clearly,

$$B_{21}\le N\int_{Z^2}(\bar{K}^2)^2\pi^2(dz)\|\phi\|_0^2,$$

and applying Hölder's inequality,

$$B_{22}\le N\int_{\mathbf{R}^{d_1}}\left(\int_{Z^2}|\phi(\tilde{\zeta}^2)-\phi|^2\pi^2(dz)\right)^{\frac12}\left(\int_{Z^2}\left(|\phi|^2+|\phi(\tilde{\zeta}^2)|^2\right)(\bar{K}^2)^2\pi^2(dz)\right)^{\frac12}dx.$$

Hence, by Remark 4.3.5 and Young's inequality,

$$B_2\le -\frac12\int_{\mathbf{R}^{d_1}}\int_{Z^2}|\phi(\tilde{\zeta}^2)-\phi|^2\pi^2(dz)dx+N\|\phi\|_0^2.$$

By Hölder's inequality,

$$B_3\le N\int_{\mathbf{R}^{d_1}}\left(\int_{Z^2}|\phi(\tilde{\zeta}^2)-\phi|^2\pi^2(dz)\right)^{\frac12}\left(\int_{Z^2}(l^2)^2\pi^2(dz)\right)^{\frac12}|\phi|dx.$$

Applying Young's inequality again and combining B_2 and B_3, we derive

$$2\langle\phi,\mathcal{L}^2\phi\rangle_1\le N\|\phi\|_0^2-\frac14\int\partial_i\phi^l\sigma^{2;i\varrho}\sigma^{2;j\varrho}\partial_j\phi^l dx-\frac14\int\int_{Z^2}|\phi(\tilde{\zeta}^2)-\phi|^2\pi^2(dz)dx. \quad (4.3.8)$$

By the divergence theorem, we have

$$2\langle\phi,b^i\partial_i\phi+c^{\cdot\bar{l}}\phi^{\bar{l}}+f\rangle_0=2\,(\phi,f)_0+(\phi,\phi\operatorname{div}b)_0+2(\phi,c\phi)_0\le N(\|\phi\|_0^2+\|f\|_0^2). \quad (4.3.9)$$

Combining (4.3.8) and (4.3.9), we obtain (4.3.6). To obtain the estimate (4.3.7), we use (4.3.6) and (4.3.7), and estimate the additional terms:

$$D:=\left(\sigma^{1;i\varrho}\partial_i\phi+\upsilon^{1;\cdot\bar{l}\varrho}\phi^{\bar{l}},g^\varrho\right)_0$$

and

$$2\int_{Z^1}\left(\left(\phi(\tilde{\zeta}^1)-\phi,h\right)_0+(\rho^1\phi(\tilde{\zeta}^1),h)_0\right)\pi^1(dz)=:E_1+E_2.$$

By the divergence theorem and Hölder's inequality,$|D|\le N\left(\|\phi\|_0^2+\|g\|_1^2\right)$. Applying Hölder's inequality and changing the variable of integration, we get

$$E_2\le\int_{\mathbf{R}^{d_1}}\int_{Z^1}\left(|\rho^1\phi(\tilde{\zeta}^1)|^2+|h|^2\right)\pi^1(dz)dx\le N\left(\|\phi\|_0^2+\int_{Z^1}\|h\,(z)\,\|_0^2\pi^1(dz)\right).$$

Then by (4.3.4), Hölder's inequality, and Lemma 4.3.10,

$$
\begin{aligned}
E_1 &= 2\int_{Z^1}\int_{\mathbf{R}^{d_1}} \phi^l \left(h^l(\tilde{\zeta}^{1;-1})(\det \nabla \tilde{\zeta}^{1;-1} - 1) + h^l(\tilde{\zeta}^{1;-1}) - h\right) dx\pi^1(dz) \\
&\le N\left(\|\phi\|_0^2 + \int_{\mathbf{R}^{d_1}}\int_{Z^1}\left(|h|^2 + |h(\tilde{\zeta}^{1;-1}) - h|^2\right)\pi^1(dz)dx\right) \\
&\le N\left(\|\phi\|_0^2 + \int_{Z^1}\|h(z)\|_{\frac{\beta}{2}}^2\pi^1(dz)\right).
\end{aligned}
$$

This completes the proof. □

In the following lemma, we verify that Assumption 4.2.2(0, 1) holds for (4.3.1). Recall that $\mathcal{W}^{0,1}$ is the space of all H^0-valued strongly càdlàg processes $v : \Omega\times[0,T] \to H^0$ that belong to $L^2(\Omega\times[0,T], \mathcal{O}_T, dV_t d\mathbf{P}; H^1)$.

Lemma 4.3.8. *Let Assumption 4.3.2(d_2) hold. Then there is a constant $N = N(d_1, d_2, N_0, \eta, \beta)$ such that for all $v \in \mathcal{W}^{0,1}$, $\mathbf{P}$-a.s.:*

(1)

$$
\begin{aligned}
&2\langle v_t, \mathcal{L}_t v_t\rangle_1 dV_t + \|\mathcal{N}_t v_t\|_0^2 dV_t + \int_{Z^1}\|\mathcal{I}_{t,z}v_{t-}\|_0^2\eta(dt,dz) \\
&\quad + 2(v_t, \mathcal{N}_t^\varrho v_t)_0 dw_t^\varrho + 2\int_{Z^1}(v_{t-}, \mathcal{I}_{t,z}v_{t-})_0\tilde{\eta}(dt,dz) \\
&\le \left(N\|v_t\|_0^2 dV_t + \int_{Z^1} N\kappa_t(z)\|v_{t-}\|_0^2\eta(dt,dz) + 2(v_t, \mathcal{N}_t^\rho v_t)_0 dw_t^\varrho + \int_{Z^1} G_{t,z}(v)\tilde{\eta}(dt,dz)\right),
\end{aligned}
$$

where

$$
|G_{t,z}(v)|dV_t \le \bar{\kappa}_t(z)\|v_{t-}\|_0^2 dV_t,\ \forall z \in Z^1, \quad |(v_t, \mathcal{N}_t v_t)_0| dV_t \le N\,\|v_t\|_0^2\, dV_t,
$$

and κ_t and $\bar{\kappa}_t$ are $\mathcal{P}_T\times\mathcal{Z}^1$-measurable processes such that for all $t \in [0,T]$,

$$
\int_{Z^1}\left(\kappa_t(z) + \bar{\kappa}_t(z)^2\right)\pi_t^1(dz) \le N;
$$

(2)

$$
\begin{aligned}
&2(\mathcal{N}_t^\varrho v_t, g_t^\varrho)_0 dV_t + 2\int_{Z^1}(\mathcal{I}_{t,z}v_{t-}, h_t(z))_0\eta(dt,dz) + 2(v_t, g_t^\varrho)_0 dw_t^\varrho + 2\int_{Z^1}(v_{t-}, h_t(z))_0\tilde{\eta}(dt,dz) \\
&\le \left(N\|v_{t-}\|_0 r_t dV_t + \int_{Z^1} N\|v_{t-}\|_0\|h_t(z)\|_0\hat{\kappa}_t(z)\eta(dt,dz) + 2(v_t, g_t^\varrho)_0 dw_t^\varrho + 2\int_{Z^1}\bar{G}_{t,z}(v)\tilde{\eta}(dt,dz)\right),
\end{aligned}
$$

where

$$r_t := \|g_t\|_1 + \left\| \int_{Z^1} \left(h_t(\tilde{\zeta}_t^{1;-1}(z), z) - h_t(z) \right) \pi_t^1(dz) \right\|_0, \ t \in [0,T],$$
$$|(v_t, g_t)_0| dV_t \leq N \|v_t\|_0 \|g_t\|_0 dV_t,$$
$$\bar{G}_{t,z}(v) dV_t \leq N \|v_{t-}\|_0 \|h_t(z)\|_0, dV_t, \ \forall z \in Z^1,$$

and $\hat{\kappa}_t$ is a $\mathcal{P}_T \times \mathcal{Z}^1$-measurable process such that for all $t \in [0,T]$,

$$\int_{Z^1} \hat{\kappa}_t(z)^2 \pi_t^1(dz) \leq N.$$

Proof. (1) Owing to the divergence theorem, we have

$$2(v_t, \mathcal{N}_t^{\varrho} v_t)_0 = (v_t, u_t \operatorname{div} \sigma_t^{1\varrho})_0 + 2(v_t, \upsilon_t^{1\varrho} v_t)_0, \quad \forall \varrho \in \mathbf{N},$$

and hence $\mathbf{P}$-a.s.,

$$|2(v_t, \mathcal{N}_t v_t)_0| dV_t \leq N \|v_t\|^2 dV_t.$$

By virtue of Lemma 4.3.7(1), it suffices to estimate

$$Q := 2\langle v_t, \mathcal{J}_{t,z}^1 v_t \rangle_1 dV_t + \int_{Z^1} \|\mathcal{I}_{t,z} v_{t-}\|_0^2 \eta(dt,dz) + 2\int_{Z^1} (v_{t-}, \mathcal{I}_{t,z} v_{t-})_0 \tilde{\eta}(dt,dz)\,.$$

An application of divergence theorem shows that

$$Q = \int_{Z^1} P_{t,z}(u) \eta(dt,dz) + \int_{Z^1} G_{t,z}(v) \tilde{\eta}(dt,dz),$$

where

$$G_{t,z}(v) := 2(v_{t-}, \rho_t^1(z) v_{t-})_0 - (v_{t-}, v_{t-} \operatorname{div} \zeta_t^1(z))_0,$$

and $P_{t,z}(v) := D_1 + D_2 + D_3$ with

$$D_1 := 2(v_{t-}(\tilde{\zeta}_t^1(z)), \rho_t^1(z) v_{t-}(\tilde{\zeta}_t^1))_0 - 2(v_{t-}, \rho_t^1(z) v_{t-})_0,$$
$$D_2 := \|v_{t-}(\tilde{\zeta}_t^1(z))\|_0^2 - \|v_{t-}\|_0^2 + (v_{t-}, v_{t-} \operatorname{div} \zeta_t^1(z))_0, \quad D_3 := \|\rho_t^1(z) v_{t-}(\tilde{\zeta}_t^1(z))\|_0^2.$$

Given our assumptions, it is clear that $\mathbf{P}$-a.s.,

$$G_{t,z}(v) dV_t \leq N \left(l_t^1(z) + \bar{K}_t^1(z) \right) \|v_{t-}\|_0^2 dV_t \quad and \quad D_3 dV_t \leq l_t^1(z)^2 \|v_{t-}\|_0^2 dV_t,$$

where in the last inequality we used the change of variable formula. Changing the variable of integration and using (4.3.4) and (4.3.5), we find that $d\mathbf{P}$ -a.s.,

$$\begin{aligned}D_1 dV_t &\leq N\left(v_{t-}, v_{t-}\left|\rho_t^1(\tilde{\zeta}_t^{1;-1}(z), z)\det\nabla\tilde{\zeta}_t^{1;-1}(z) - \rho_t^1(z)\right|\right)_0 dV_t\\ &\leq N\left(l_t^1(z)\bar{K}_t^1(z) + \tilde{l}_t^1(z)K_t^1(z)^{\frac{\beta}{2}}\right)\|v_{t-}\|_0^2 dV_t,\end{aligned}$$

and

$$D_2 dV_t = \left(v_{t-}, v_{t-}|\det\nabla\tilde{\zeta}_t^{1;-1}(z) - 1 + \operatorname{div}\zeta_t^1(z)|\right)_0 dV_t \leq \left(\bar{K}_t^1(z)^2 + \tilde{K}_t^1(z)K_t^1(z)^{\frac{\beta}{2}}\right)N\|v_{t-}\|_0^2 dV_t.$$

Setting

$$\kappa_t(z) = l_t^1(z)^2 + l_t^1\bar{K}_t^1(z) + \tilde{l}_t^1(z)K_t^1(z)^{\frac{\beta}{2}} + \bar{K}_t^1(z)^2 + \tilde{K}_t^1(z)K_t^1(z)^{\frac{\beta}{2}}, \quad \bar{\kappa}_t(z) = l_t^1(z) + \bar{K}_t^1(z),\ z \in Z^1,$$

and appealing to our assumptions, we complete the proof (1).

(2) By the divergence theorem, we have

$$(g_t^\varrho, \mathcal{N}_t^\varrho v_t)_0 = (g_t^\varrho, \operatorname{div}\sigma_t^{1\varrho} v_t)_0 + (\sigma^{1;i\varrho}\partial_i g_t^\varrho, v_t)_0, \quad \forall\rho \in \mathbf{N},$$

and thus by the Cauchy-Schwarz inequality,

$$|(g_t, \mathcal{N}_t v_t)_0| dV_t \leq N\|v_t\|_0\|g_t\|_1 dV_t.$$

Changing the variable of integration, we obtain

$$\begin{aligned}(\mathcal{I}_{t,z}v_{t-}, h_t(z))_0 &= \left(h_t(z), \left(v_{t-}(\tilde{\zeta}_t^1(z)) - v_{t-} + \rho_t^1(z)v_{t-}(\tilde{\zeta}_t^1(z))\right)\right)_0\\ &= (h_t(\tilde{\zeta}_t^{1;-1}(z), z) - h_t(z), v_{t-})_0 + (h_t(\tilde{\zeta}_t^{1;-1}(z), z), (\det\nabla\tilde{\zeta}_t^{1;-1}(z) - 1)v_{t-})_0\\ &\quad + (h_t(z), \rho_t^1(z)v_{t-}(\tilde{\zeta}_t^1(z)))_0.\end{aligned}$$

A simple calculation shows that $\mathbf{P}$-a.s.,

$$\begin{aligned}&2\int_{Z^1}(\mathcal{I}_{t,z}v_{t-}, h_t(z))_0\eta(dt,dz) + 2\int_{Z^1}(v_{t-}, h_t(z))_0\tilde{\eta}(dt,dz)\\ &\leq 2\|v_{t-}\|_0 r_t^1 dV_t + \int_{Z^1}\bar{P}_{t,z}(v)\eta(dt,dz) + \int_{Z^1}\bar{G}_{t,z}(v)\tilde{\eta}(dt,dz),\end{aligned}$$

where

$$r_t^1 := \left\|\int_{Z^1}(h_t(\tilde{\zeta}_t^{1;-1}(z), z) - h_t(z))\pi_t^1(dz)\right\|_0, \quad \bar{G}_{t,z}(v) = (h_t(\tilde{\zeta}_t^{1;-1}(z), z), v_t),$$

and

$$P_{t,z}(v) := (h_t(\tilde{\zeta}_t^{1;-1}(z)), v_{t-}(\det\nabla\tilde{\zeta}_t^{1;-1}(z) - 1))_0 + (h_t(z), \rho_t^1(z)v_{t-}(\tilde{\zeta}_t^1(z)))_0.$$

Applying the change of variable formula and Hölder's inequality, $\mathbf{P}$-a.s. we obtain

$$\int_{Z^1} \tilde{P}_{t,z}(u)\eta(dt,dz) \leq N\|v_{t-}\|_0 \int_{Z^1} (\bar{K}_t^1(z) + l_t^1(z))\|h_t(z)\|_0\eta(dt,dz)$$

and

$$|\tilde{G}_{t,z}(v)|dV_t \leq N\|v_{t-}\|_0\|h_t(z)\|_0 dV_t.$$

This completes the proof. □

Let $d \in \mathbf{N}$. For a function $v \in H^m(\mathbf{R}^{d_1}, \mathbf{R}^d)$, define the linear operator $\mathcal{D}v \in H^{m-1}(\mathbf{R}^{d_1}; \mathbf{R}^{d(d_1+1)})$ by

$$\mathcal{D}v = (\partial_0 v, \partial_1 v, \ldots, \partial_{d_1} v) = \tilde{v}$$

with $\tilde{v}^{l0} = v^l$ and $\tilde{v}^{lj} = \partial_j v^l, 1 \leq l \leq d, 0 \leq j \leq d_1$ (recall $\partial_0 v = v$). We define $\mathcal{D}^n v$ for $n \in \mathbf{N}$ by iteratively applying $\mathcal{D}$ n-times. Recall that $\Lambda = (I - \Delta)^{\frac{1}{2}}$. It is easy to check that for each $n \in \mathbf{N}$ and all $u, v \in H^{n+1}(\mathbf{R}^{d_1}, \mathbf{R}^d)$,

$$\begin{aligned} (u, v)_{n,d} &= (\Lambda^n u, \Lambda^n v)_{0,d} = (\mathcal{D}^n u, \mathcal{D}^n v)_{0,d\bar{d}_1^n}, \\ (\Lambda u, \Lambda^{-1} v)_{n,d} &= (\Lambda^{n+1} u, \Lambda^{n-1} v)_{0,d} = (\mathcal{D}^n \Lambda u, \mathcal{D}^n \Lambda^{-1} v)_{0,d\bar{d}_1^n}, \\ (\mathcal{D}^n u, \mathcal{D}^n v)_{-1,d\bar{d}_1^n} &= (\mathcal{D}^n \Lambda^{-1} u, \mathcal{D}^n \Lambda^{-1} v)_{0,d\bar{d}_1^n} = (u, v)_{n-1,d}, \end{aligned} \tag{4.3.10}$$

where $\bar{d}_1 = d_1 + 1$. Let us introduce the operators $\mathcal{E}(\mathcal{L})$, $\mathcal{E}(\mathcal{N})$, and $\mathcal{E}(\mathcal{I}_z)$ acting on $\phi = (\phi^{lj})_{1\leq l\leq d_2, 1\leq j\leq \bar{d}_1} \in C_c^\infty(\mathbf{R}^{d_1}, \mathbf{R}^{d_2\bar{d}_1})$ that are defined as $\mathcal{L}, \mathcal{N}$, and $\mathcal{I}$, respectively, but with the $d_2 \times d_2$-dimensional coefficients υ_t^k, ρ^k, and c replaced by the $d_2\bar{d}_1 \times d_2\bar{d}_1$-dimensional coefficients given by

$$\begin{aligned} \upsilon^{k;lj,\bar{l}\bar{j}\varrho} &= \upsilon^{k;l\bar{l}\varrho}\delta_{j\bar{j}} + 1_{j\geq 1}(\partial_j \sigma^{k;\bar{j}\varrho}\delta_{l\bar{l}} + \partial_j \upsilon^{k;l\bar{l}\varrho}\delta_{\bar{j}0}), \\ \rho^{k;lj,\bar{l}\bar{j}} &= \rho_t^{k;l\bar{l}}\delta_{\bar{j}j} + 1_{j\geq 1}(\partial_j \rho^{k;l\bar{l}}\delta_{\bar{j}0} + (\delta_{l\bar{l}} + \rho^{k;l\bar{l}})\partial_j \zeta^{k;\bar{j}}), \end{aligned}$$

and

$$c^{lj,\bar{l}\bar{j}} = c^{l\bar{l}}\delta_{\bar{j}j} + \partial_j b^{\bar{j}}\delta_{l\bar{l}} + \partial_j c^{l\bar{l}}\delta_{\bar{j}0} + \sum_{k=1}^{2}\left(\upsilon^{k;l\bar{l}\varrho}\partial_j\sigma^{k;\bar{j}\varrho} + \int_{Z^k} \rho^{k;l\bar{l}}\partial_j\zeta^{k;\bar{j}}\pi_t^k(dz)\right),$$

for $1 \leq l, \bar{l} \leq d_2$ and $0 \leq j, \bar{j} \leq d_1$. The coefficients σ^k, b, and functions $\zeta^k, k \in \{1, 2\}$, remain unchanged in the definition of $\mathcal{E}(\mathcal{L})$, $\mathcal{E}(\mathcal{N})$, and $\mathcal{E}(\mathcal{I})$. We define $\mathcal{E}^n(\mathcal{L})$, $\mathcal{E}^n(\mathcal{N})$, and $\mathcal{E}^n(\mathcal{I})$, for $n \in \mathbf{N}$ by iteratively applying $\mathcal{E}$ n-times by the rules above with σ^k, b, and $\zeta^k, k \in \{1, 2\}$, unchanged. A simple calculation shows that for all $v \in H^2(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$,

$$\mathcal{D}[\mathcal{L}v] = \mathcal{E}(\mathcal{L})\mathcal{D}v, \quad \mathcal{D}[\mathcal{N}^\varrho v] = \mathcal{E}(\mathcal{N}^\varrho)\mathcal{D}v, \ \varrho \in \mathbf{N}, \quad \mathcal{D}[\mathcal{I}_z v] = \mathcal{E}(\mathcal{I}_z)\mathcal{D}v.$$

Continuing, for all $v \in H^{n+1}(\mathbf{R}^{d_1};\mathbf{R}^d)$ we have

$$\mathcal{D}^n[\mathcal{L}v] = \mathcal{E}^n(\mathcal{L})\mathcal{D}^n v, \quad \mathcal{D}^n[\mathcal{N}^\varrho v] = \mathcal{E}^n(\mathcal{N}^\varrho)\mathcal{D}^n v, \ \varrho \in \mathbf{N}, \quad \mathcal{D}^n[\mathcal{I}_z v] = \mathcal{E}^n(\mathcal{I}_z)\mathcal{D}^n v. \tag{4.3.11}$$

If Assumption 4.3.1(m,d_2) holds, it can readily be verified by induction and the definitions (4.3.2)-(4.3.2) that Assumption 4.3.2$(0,d_2\bar{d}_1^m)$ holds for the coefficients of the operators $\mathcal{E}^m(\mathcal{L})$, $\mathcal{E}^m(\mathcal{N})$, and $\mathcal{E}^m(\mathcal{I})$. Moreover, owing to our assumptions on the input data, we have

$$\mathcal{D}^m\phi \in \mathbf{H}^0(\mathbf{R}^{d_1};\mathbf{R}^{d_2\bar{d}_1^m};\mathcal{F}_0), \quad \mathcal{D}^m f \in \mathbf{H}^0(\mathbf{R}^{d_1};\mathbf{R}^{d_2\bar{d}_1^m})$$
$$\mathcal{D}^m g \in \zeta^1(\mathbf{R}^{d_1};\ell_2(\mathbf{R}^{d_2\bar{d}_1^m})), \quad \mathcal{D}^m h \in \mathbf{H}^{\frac{\beta}{2}}(\mathbf{R}^{d_1};\mathbf{R}^{d_2\bar{d}_1^m};\pi^1).$$

Making use of (4.3.10), (4.3.11) and applying Lemma 4.3.6 to $\mathcal{E}^m(\mathcal{L})$, for all $v \in H^{m+1}$, we obtain

$$\|\mathcal{L}v\|_{m-1} = \|\mathcal{D}^m[\mathcal{L}v]\|_{-1} = \|\mathcal{E}^m(\mathcal{L})\mathcal{D}^m v\|_{-1,d_2\bar{d}_1^m} \le N\|\mathcal{D}^m v\|_{1,d_2\bar{d}_1^m} = N\|v\|_{m+1}.$$

Likewise, for all $v \in H^{m+1}$, we derive

$$\|\mathcal{N}v\|_m \le N\|v\|_{m+1}, \qquad \int_{Z^1} \|\mathcal{I}v\|_m^2 \pi^1(dz) \le N\|v\|_{m+1}^2.$$

By virtue of Lemma 4.3.7, we have that for all $v \in H^{m+1}$,

$$2(\Lambda v, \Lambda^{-1}\mathcal{L}_t v)_m + \|\mathcal{N}_t v\|_m^2 + \int_{Z^1} \|\mathcal{I}_{t,z}v\|_m^2 \pi^1\,(dz)$$
$$= 2(\mathcal{D}^m\Lambda v, \mathcal{D}^m\Lambda^{-1}[\mathcal{L}_t v])_0 + \|\mathcal{D}^n[\mathcal{N}_t v]\|_0^2 + \int_{Z^1} \|\mathcal{D}^m[\mathcal{I}_{t,z}v]\|_0^2 \pi^1(dz)$$
$$2\langle \mathcal{D}^m v, \mathcal{E}^m(\mathcal{L}_t)\mathcal{D}^m v\rangle_1 + \|\mathcal{E}^m(\mathcal{N}_t)\mathcal{D}^m v\|_0^2 + \int_{Z^1} \|\mathcal{E}^m(\mathcal{I}_{t,z})\mathcal{D}^n v\|_0^2 \pi^1(dz) \le N\,\|\mathcal{D}^m v\|_{0,d_2\bar{d}_1^m}^2 = N\,\|v\|_m^2$$

Using a similar argument, we find that for all $v \in H^{m+1}$,

$$2(\Lambda v, \Lambda^{-1}(\mathcal{L}_t v + f_t))_m + \|\mathcal{N}_t v + g_t\|_m^2 + \int_{Z^1} \|\mathcal{I}_{t,z}v + h_t(z)\|_m^2 \pi^1(dz) \le N\|v\|_m^2 + N\bar{f}_t,$$

where

$$\bar{f}_t = \|f_t\|_m^2 + \|g_t\|_{m+1}^2 + \int_{Z^1} \|h_t(z)\|_{m+\frac{\beta}{2}}^2 \pi_t^1(dz).$$

Therefore, Assumption 4.2.1(m,d_2) holds for the equation (4.3.1). Similarly, using Lemmas 4.3.8 and 4.3.10, we find that Assumption 4.2.2(m,d_2) holds for equation (4.3.1) as well. The statement of the theorem then follows directly from Theorem 4.2.4.

4.3.3 Appendix

For each $\kappa \in (0,1)$ and tempered distribution f on $\mathbf{R}^{d_1}$, we define

$$\partial^\kappa f = \mathcal{F}^{-1}[|\cdot|^\kappa \mathcal{F} f(\cdot)],$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

Lemma 4.3.9 (cf. Lemma 2.1 in [Kom84])**.** *Let $f : \mathbf{R}^{d_1} \to \mathbf{R}$ be smooth and bounded. Then for all $\kappa \in (0,1)$, there are constants $N_1 = N_1(d_1,\kappa)$, $N_2 = N_2(d_1,\kappa)$, and $N_3 = N_2(d_1,\kappa)$ such that for all $x, y, z \in \mathbf{R}^{d_1}$,*

$$\partial^\kappa f(x) = N_1 \int_{\mathbf{R}^d} (f(x+z) - f(x)) \frac{dz}{|z|^{d+\delta}}$$

and

$$f(x+y) - f(x) = N_2 \int_{\mathbf{R}^{d_1}} \partial^\kappa f(x-z) k^{(\kappa)}(y,z) dz,$$

where

$$k^{(\kappa)}(y,z) = |y+z|^{\kappa-d} - |z|^{\kappa-d} \quad \text{and} \quad \int_{\mathbf{R}^{d_1}} |k^{(\kappa)}(y,z)| dz = N_3 |z|^\kappa.$$

Lemma 4.3.10. *Let $(Z, \mathcal{Z}, \pi)$ be a sigma-finite measure space. Let $H : \mathbf{R}^{d_1} \times Z \to \mathbf{R}^{d_1}$ be $\mathcal{B}(\mathbf{R}^d) \otimes \mathcal{Z}$-measurable and assume that for all $(x,z) \in \mathbf{R}^{d_1} \times Z$,*

$$|\zeta(x,z)| \leq K(z) \quad \text{and} \quad |\nabla \zeta(x,z)| \leq \bar{K}(z)$$

where $K, \bar{K} : Z \to \mathbf{R}_+$ is a $\mathcal{Z}$-measurable function for which there is a positive constant N_0 such that for some fixed $\beta \in (0,2]$,

$$\sup_{z \in Z} K(z) + \sup_{z \in Z} \bar{K}(z) + \int_Z \left(K(z)^\beta + \bar{K}(z)^2 \right) \pi(dz) < N_0$$

Assume that there is a constant $\eta < 1$ such that $(x,z) \in \{(x,z) \in \mathbf{R}^{d_1} \times Z : |\nabla \zeta(x,z)| > \eta\}$,

$$|(I_{d_1} + \nabla \zeta_t(x,z))^{-1}| \leq N_0.$$

Then there is a constant $N = N(d_1, N_0, \beta, \eta)$ such that for all $\mathcal{B}(\mathbf{R}^{d_1}) \otimes \mathcal{Z}$-measurable $h : \mathbf{R}^{d_1} \times Z \to \mathbf{R}^{d_2}$ with $h \in L^2(Z, \mathcal{Z}, \pi; H^{\frac{\beta}{2}}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))$,

$$\int_{\mathbf{R}^{d_1}} \left| \int_Z (h(x+\zeta(x,z),z) - h(x,z)) \pi(dz) \right|^2 dx \leq N \int_Z \|h(z)\|^2_{\frac{\beta}{2}} \pi(dz).$$

Proof. It is easy to see that for any $\mathcal{B}(\mathbf{R}^{d_1}) \otimes \mathcal{Z}$-measurable $h : \mathbf{R}^{d_1} \times Z \to \mathbf{R}^{d_2}$ such that

$$\int_Z \sup_{x \in \mathbf{R}^{d_1}} |\nabla h(x,z)|^2 \pi(dz) < \infty, \tag{4.3.12}$$

the integral $\int_Z (h(x+\zeta(x,z),z) - h(x,z))\pi(dz)$ is well-defined. Moreover, for any $\mathcal{B}(\mathbf{R}^{d_1}) \otimes \mathcal{Z}$-measurable $h : \mathbf{R}^{d_1} \times Z \to \mathbf{R}$ with $h \in L^2(Z, \mathcal{Z}, \pi; H^{\frac{\beta}{2}}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))$, we can always find a sequence $(h^n)_{n \in \mathbf{N}}$ of $\mathcal{B}(\mathbf{R}^{d_1}) \otimes \mathcal{Z}$-measurable processes such that each element of the sequence is smooth with compact support in x and satisfies (4.3.12) and

$$\lim_{n \to \infty} \int_Z \|h(z) - h^n(z)\|^2_{\frac{\beta}{2}} \pi(dz) = 0.$$

Thus, if we prove this lemma for h that is smooth with compact support in x and satisfies (4.3.12), then we can conclude that the sequence

$$\int_Z (h^n(x + \zeta(x,z), z) - h^n(x,z))\, \pi(dz), \quad n \in \mathbf{N},$$

is Cauchy in $H^0(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$. We then define

$$\int_Z (h(x + \zeta(x,z), z) - h(x,z))\, \pi(dz)$$

for any $\mathcal{B}(\mathbf{R}^{d_1}) \otimes \mathcal{Z}$-measurable $h : \mathbf{R}^{d_1} \times Z \to \mathbf{R}$ with $h \in L^2(Z, \mathcal{Z}, \pi; H^{\frac{\beta}{2}}(\mathbf{R}^{d_1}; \mathbf{R}^{d_2}))$ to be the unique $H^0(\mathbf{R}^{d_1}; \mathbf{R}^{d_2})$ limit of the Cauchy sequence. Hence, it suffices to consider h that is smooth with compact support in x and satisfies (4.3.12). First, let us consider the case $\beta \in (0,2)$. By Lemma 4.3.9, we have

$$\begin{aligned}
&\int_{\mathbf{R}^{d_1}} \left| \int_Z \Big(h(\tilde{\zeta}(x,z), z) - h(x,z) \Big) \pi(dz) \right|^2 dx \\
&= N_2^2 \int_{\mathbf{R}^{d_1}} \left| \int_Z \int_{\mathbf{R}^{d_1}} \partial^{\frac{\beta}{2}} h(x-y,z) k^{(\frac{\beta}{2})}(\zeta(x,z), y)\, dy \pi(dz) \right|^2 dx \\
&=: N_2^2 \int_{\mathbf{R}^{d_1}} | \int_Z A(x,z) \pi(dz)|^2 dx.
\end{aligned}$$

Applying Hölder's inequality and Lemma 4.3.9, for all x and z, we have

$$\begin{aligned}
A(x,z) &\le \left(\int_{\mathbf{R}^{d_1}} |\partial^{\beta/2} h(x-y,z)|^2 k^{(\frac{\beta}{2})}(\zeta(x,z), y))\, dy \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^{d_1}} k^{(\frac{\beta}{2})}(\zeta(x,z), y))\, dy \right)^{\frac{1}{2}} \\
&= \sqrt{N_3} \left(\int_{\mathbf{R}^{d_1}} |\partial^{\beta/2} h(x-y,z)|^2 k^{(\frac{\beta}{2})}(\zeta(x,z), y)\, dy \right)^{\frac{1}{2}} |\zeta_t(x,z)|^{\frac{\beta}{4}}
\end{aligned}$$

$$\leq K(z)^{\frac{\beta}{2}}\sqrt{N_3}\left(\int_{\mathbf{R}^{d_1}}|\partial^{\frac{\beta}{2}}h(x-y,z)|^2k^{(\frac{\beta}{2})}(\zeta(x,z),y)\,dyK(z)^{-\frac{\beta}{2}}\right)^{\frac{1}{2}}.$$

Using Hölder's inequality again, for all x, we get

$$\left|\int_Z A(x,z)\pi(dz)\right|^2 \leq N_3N_0\int_Z\int_{\mathbf{R}^{d_1}}|\partial^{\beta/2}h(x-y,z)|^2k^{(\frac{\beta}{2})}(\zeta(x,z),y)\,dyK(z)^{-\frac{\beta}{2}}\pi(dz).$$

For each x and z, we set

$$B(x,z) = \int_{|y|\leq 2K(z)}|\partial^{\frac{\beta}{2}}h(x-y,z)|^2k^{(\frac{\beta}{2})}(\zeta(x,z),y)\,dy$$
$$C(x,z) = \int_{|y|>2K(z)}|\partial^{\frac{\beta}{2}}h(x-y,z)|^2k^{(\frac{\beta}{2})}(\zeta(x,z),y)\,dy.$$

Changing the variable integration, for all x and z, we find

$$\begin{aligned} B(x,z) &\leq \int_{|y+\zeta(x,z)|\leq 3K(z)}|\partial^{\frac{\beta}{2}}h(x-y,z)|^2\frac{dy}{|y+\zeta(x,z)|^{d_1-\frac{\beta}{2}}}\\ &\quad+\int_{|y|\leq 2K(z)}|\partial^{\frac{\beta}{2}}h(x-y,z)|^2\frac{dy}{|y|^{d_1-\frac{\beta}{2}}} =: B_1(x,z)+B_2(x,z), \end{aligned}$$

and

$$\begin{aligned} B_1(x,z) &\leq \int_{|y|\leq 3K(z)}|\partial^{\frac{\beta}{2}}h((\tilde{\zeta}(x,z)-y),z)|^2|\frac{dy}{|y|^{d_1-\frac{\beta}{2}}}\\ &\leq K(z)^{\frac{\beta}{2}}\int_{|y|\leq 3}|\partial^{\frac{\beta}{2}}h((\tilde{\zeta}(x,z)-yK(z)),z)|^2\frac{dy}{|y|^{d_1-\frac{\beta}{2}}},\\ B_2(x,z) &\leq K(z)^{\frac{\beta}{2}}\int_{|y|\leq 2}|\partial^{\beta/2}h(x-yK(z),z)|^2\frac{dy}{|y|^{d_1-\frac{\beta}{2}}}. \end{aligned}$$

Owing to Remark 4.3.5, for all z, the map $x\mapsto x+\zeta(x,z)=\tilde{\zeta}(x,z)$ is a global diffeomorphism and

$$\det\nabla\tilde{\zeta}^{-1}(x,z)\leq N.$$

for some constant $N=N(N_0,d_1,\eta)$. Thus, by the change of variable formula, there is a constant $N=N(d_1,N_0,\beta,\eta)$ such that

$$\begin{aligned} &\int_{\mathbf{R}^{d_1}}\int_Z B_1(x,z)K(z)^{-\frac{\beta}{2}}\pi(dz)dx\\ &\leq \int_Z\int_{|y|\leq 3}\int_{\mathbf{R}^{d_1}}|\partial^{\beta/2}h((\tilde{\zeta}(x,z)-yK(z)),z)|^2dx\frac{dy}{|y|^{d_1-\frac{\beta}{2}}}\pi(dz)\\ &\leq \int_Z\int_{|y|\leq 3}\int_{\mathbf{R}^{d_1}}|\partial^{\beta/2}h((x-yK(z)),z)|^2|\det\nabla\tilde{\zeta}^{-1}(x,z)|dx\frac{dy}{|y|^{d_1-\frac{\beta}{2}}}\pi(dz) \end{aligned}$$

$$\leq N \int_Z \int_{\mathbf{R}^{d_1}} |\partial^{\frac{\beta}{2}} h(x,z)|^2 dx\pi(dz)$$

and

$$\int_{\mathbf{R}^{d_1}} \int_Z K(z)^{\frac{\beta}{2}} B_2(x,z) dx\pi(dz) \leq N \int_Z \int_{\mathbf{R}^{d_1}} |\partial^{\frac{\beta}{2}} h(x,z)|^2 dx\pi(dz).$$

For all x, y, and z such that $|\zeta(x,z)| \leq K(z) \leq \frac{1}{2}|y|$, we have

$$\begin{aligned}
&\left| \frac{1}{|y+\zeta(x,z)|^{d_1-\frac{\beta}{2}}} - \frac{1}{|y|^{d_1-\beta/2}} \right| \\
&\leq \left| d_1 - \frac{\beta}{2} \right| \left| \left(\frac{1}{|y+\zeta(x,z)|^{1+d_1-\frac{\beta}{2}}} + \frac{1}{|y|^{1+d_1-\frac{\beta}{2}}} \right) \right| |\zeta(x,z)| \leq 3 \left| d_1 - \frac{\beta}{2} \right| \frac{|\zeta(x,z)|}{|y|^{1+d_1-\frac{\beta}{2}}},
\end{aligned}$$

and hence for all x and z,

$$\begin{aligned}
C(x,z) &= \int_{|y|>2K(z)} |\partial^{\frac{\beta}{2}} h(x-y,z)|^2 \left| \frac{1}{|y+\zeta(x,z)|^{d-\frac{\beta}{2}}} - \frac{1}{|y|^{d-\frac{\beta}{2}}} \right| dy \\
&\leq N \int_{|y|>2K(z)} |\partial^{\frac{\beta}{2}} h(x-y,z)|^2 \frac{|K(z)|}{|y|^{1+d_1-\frac{\beta}{2}}} dy \\
&\leq N K(z)^{\frac{\beta}{2}} \int_{|y|>2} |\partial^{\frac{\beta}{2}} h((x-K(z)y),z)|^2 \frac{dy}{|y|^{1+d_1-\frac{\beta}{2}}}.
\end{aligned}$$

Estimating as above, we find that there is a constant $N = N(d_1, N_0, \beta)$

$$\int_Z \int_{\mathbf{R}^{d_1}} K(z)^{\frac{\beta}{2}} C(x,z) dx\pi(dz) \leq N \int_Z \int_{\mathbf{R}^{d_1}} |\partial^{\frac{\beta}{2}} h(x,z)|^2 dx\pi(dz).$$

Combining the above estimates, we obtain the desired estimate for $\beta \in (0,2)$. Let us now consider the case $\beta = 2$. It follows from Remark 4.3.5 that for all $\theta \in [0,1]$, on the set of $z \in \{z : \bar{K}(z) < \frac{1}{2}\}$, the map $x \mapsto x + \theta\zeta(x,z) = \tilde{\zeta}_\theta(x,z)$ is a global diffeomorphism and

$$\det \nabla \tilde{\zeta}_\theta^{-1}(x,z) \leq N,$$

for some constant $N = N(N_0, d_1)$. Hence, making use of Taylor's theorem and the change of variable formula, we find

$$\begin{aligned}
&\int_{\mathbf{R}^{d_1}} \left| \int_Z \left(h(x+\zeta(x,z),z) - h(x,z) \right) \pi(dz) \right|^2 dx \\
&\leq \int_{\mathbf{R}^{d_1}} \left| \int_{\bar{K}(z)\geq\frac{1}{2}} \left(h(x+\zeta(x,z),z) - h(x,z) \right) \pi(dz) \right|^2 dx \\
&+ \int_{\mathbf{R}^{d_1}} \left| \int_{\bar{K}(z)<\frac{1}{2}} \int_0^1 |\nabla h(x+\theta\zeta(x,z),z)| \, d\theta K(z) \pi(dz) \right|^2 dx
\end{aligned}$$

$$\leq \pi\left\{\bar{K}(z) \geq \frac{1}{2}\right\} \int_{\bar{K}(z)\geq\eta} \int_{\mathbf{R}^{d_1}} |h(x,z)|^2 |\det \tilde{\zeta}^{-1}(x,z) + 1| dx\pi(dz)$$

$$+N_0 \int_{\bar{K}(z)<\frac{1}{2}} \int_{\mathbf{R}^{d_1}} \int_0^1 |\nabla h(x,z)|^2 |\det \nabla \tilde{\zeta}_\theta^{-1}(x,z)| d\theta dx\pi(dz) \leq N \int_Z \|h(z)\|_1^2 \pi(dz).$$

This completes the proof. □

Chapter 5

A finite difference scheme for non-degenerate parabolic SIDEs

5.1 Introduction

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$, $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$, be a complete filtered probability space such that the filtration is right continuous and $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets of $\mathcal{F}$. Let w_t^ϱ, $t \geq 0$, $\varrho \in \mathbf{N}$, be a sequence of independent real-valued $\mathbf{F}$-adapted Wiener processes. Let $\pi^1(dz)$ and $\pi^2(dz)$ be a Borel sigma-finite measures on $\mathbf{R}^d$ satisfying

$$\int_{\mathbf{R}^d} |z|^2 \wedge 1 \; \pi_r(dz) < \infty, \quad r \in \{1, 2\}.$$

Let $q(dt, dz) = p(dt, dz) - \pi^2(dz)dt$ be a compensated $\mathbf{F}$-adapted Poisson random measure on $\mathbf{R}_+ \times \mathbf{R}^d$. Let $T > 0$ be an arbitrary fixed constant. On $[0, T] \times \mathbf{R}^d$, we consider finite difference approximations for the following SIDE

$$du_t = ((\mathcal{L}_t + I)u_t + f_t)\, dt + \sum_{\varrho=1}^{\infty} (\mathcal{N}_t^\varrho u_t + g_t^\varrho)\, dw_t^\varrho + \int_{\mathbf{R}^d} (\mathcal{I}(z)u_{t-} + o_t(z))\, q(dt, dz), \tag{5.1.1}$$

with initial condition

$$u_0(x) = \varphi(x), \quad x \in \mathbf{R}^d,$$

where the operators are given by

$$\mathcal{L}_t\phi(x) := \sum_{i,j=0}^{d} a_t^{ij}(x)\partial_{ij}\phi(x),$$

$$I\phi(x) := \int_{\mathbf{R}^d} \left(\phi(x+z) - \phi(x) - \mathbf{1}_{[-1,1]}(|z|) \sum_{j=1}^{d} z_j \partial_j \phi(x) \right) \pi^1(dz), \tag{5.1.2}$$

$$\mathcal{N}_t^\varrho \phi(x) := \sum_{i=0}^{d} \sigma_t^{i\varrho}(x)\partial_i \phi(x), \quad \mathcal{I}(z)\phi(x) = \phi(x+z) - \phi(x).$$

Here, we denote the identity operator by ∂_0.

Equation (5.1.1) arises naturally in non-linear filtering of jump-diffusion processes. We refer the reader to [Gri82] and [GM11] for more information about non-linear filtering of jump-diffusions and the derivation of the Zakai equation. Various methods have been developed to solve stochastic partial differential equations (SPDEs) numerically. For SPDEs driven by continuous martingale noise see, for example, [GK96, Gyö98, Gyö99, GM09, LR04, JK10, Yan05], and for SPDEs driven by discontinuous martingale noise, see [HM06, Hau08, Lan12, BL12]. Among the various methods considered in the literature is the method of finite differences. For second order linear SPDEs driven by continuous martingale noise it is well-known that the $L^p(\Omega)$-pointwise error of approximation in space is proportional to the parameter h of the finite difference (see, e.g., [Yoo00]). In [GM09], I. Gyöngy and A. Millet consider abstract discretization schemes for stochastic evolution equations driven by continuous martingale noise in the variational framework and, as a particular example, show that the $L^2(\Omega)$-pointwise rate of convergence of an Euler-Maruyuma (explicit and implicit) finite difference scheme is of order one in space and one-half in time. More recently, it was shown by I. Gyöngy and N.V. Krylov that under certain regularity conditions, the rate of convergence in space of a semi-discretized finite difference approximation of a linear second order SPDE driven by continuous martingale noise can be accelerated to any order by Richardson's extrapolation method. For the non-degenerate case, we refer to [GK10] and [GK11], and for the degenerate case, we refer to [Gyö11]. In [Hal12] and [Hal13], E. Hall proved that the same method of acceleration can be applied to implicit time-discretized SPDEs driven by continuous martingale noise. Also, for a pathwise convergence result of a spectral scheme for SPDEs on a bounded domain we refer the reader to [CJM92].

In the literature, finite element, spectral, and, more generally, Galerkin schemes have been studied for SPDEs driven by discontinuous martingale noise. One of the earliest works in this direction is a paper [HM06] by E. Hausenblas and I. Marchis concerning $L^p(\Omega)$-convergence of Galerkin approximation schemes for abstract stochastic evolution equations in Banach spaces driven by Poisson noise of impulsive-type. As an application of their result, they study a spectral approximation of a linear SPDE in $L^2([0,1])$ with Neumann boundary conditions driven by Poisson noise of impulsive-type and derive $L^p(\Omega)$-error estimates in the $L^2([0,1])$-norm. In [Hau08], E. Hausenblas considers finite element approximations of linear SPDEs in polyhedral domains D driven by Poisson noise of impulsive-type and derives $L^p(\Omega)$ error estimates in the $L^p(D)$-norm. In a more recent work [Lan12], A. Lang studied semi-discrete Galerkin approximation schemes for SPDEs of advection diffusion type in bounded domains D driven by cádlág square integrable martingales in a Hilbert Space. A. Lang showed that the rate of convergence in the $L^p(\Omega)$ and almost-sure sense in the $L^2(D)$-norm is of order two for a finite-element Galerkin scheme. In [BL12], A. Lang and A. Barth derive $L^2(\Omega)$ and almost-sure estimates in the $L^2(D)$-

norm for the error of a Milstein-Galerkin approximation scheme for the same equation considered in [Lan12] and obtain convergence of order two in space and order one in time.

In the articles [Lan12, BL12, HM06, Hau08], the authors make use of the semigroup theory of SPDEs (mild solution) and only consider SPDEs in which the principal part of the operator in the drift is non-random. Moreover, the authors there do not address the approximation of equations with non-local operators in the drift or noise. The principal part of the operator in the drift of the Zakai equation is, in general, random, and hence numerical schemes that approximate SPDEs or SIDEs with random-adapted principal part are of importance. More precisely, the coefficients of the Zakai equation are random if the coefficients of the SDE governing the signal depend on the observation. In this chapter, since we use the variational framework (L^2-theory) of SPDEs, we are easily able to treat the case of random-coefficients, and hence the diffusion coefficients $a_t^{ij}(x)$ appearing in (5.1.1) are random.

In dimension one, a finite difference scheme for degenerate integro-differential equations (deterministic) has been studied by R. Cont and E. Voltchkova in [CV05]. The authors in [CV05] first approximate the integral operator near the origin with a second derivative operator. The resulting PDE is then non-degenerate and has an integral operator of order zero. The error of this approximation is obtained by means of the probabilistic representation of the solution of both the original equation and the non-degenerate equation. In the second step of their approximation, R. Cont and E. Voltchkova consider an implicit-explicit finite difference scheme and obtain pointwise error estimates of order one in space. As a consequence of the two-step approximation scheme, there are two separate errors for the approximation

In this chapter, we consider the non-degenerate stochastic integro-differential equation (5.1.1) with random coefficients and apply the method of finite differences in the time and space variables. To the best of our knowledge, this article is the first to use the finite difference method to approximate SIDEs. The approximations of the non-local integral operators in the drift and in the noise of (5.1.1) we choose are both natural and relatively easy to implement. In particular, we are able to treat the singularity of the integral operators near the origin directly. We consider a fully-explicit time-discretization scheme and an implicit-explicit time-discretization scheme, where we treat part of the approximation of the integral operator in the drift explicitly.

To obtain error estimates for our approximations, we use the approach in [Yoo00], where the discretized equations are first solved as time-discretized SDEs in Sobolev spaces over $\mathbf{R}^d$ and an error estimate is obtained in Sobolev norms. After obtaining $L^2(\Omega)$ error estimates in Sobolev norms, the Sobolev embedding theorem is used to obtain $L^2(\Omega)$-pointwise error estimates. So, in sum, we obtain two types of error estimates: in Sobolev norms and on the grid. Naturally, when using the Sobolev embedding to obtain the pointwise estimates, we do not need the equation to be differentiable to obtain pointwise error estimates, only continuous. Using the approach of first obtaining estimates in Sobolev spaces, we are also easily able to deduce that the more regularity on the coefficients and data we have, the stronger the error estimates we can obtain (see Corollaries 5.5.3 and 5.5.4).

The chapter is organized as follows. In the next section, we introduce the notation that will be used throughout the chapter and state the main results. In the third section, we present a simulation that we did to confirm our rates of convergence. In the fourth section, we prove auxiliary results that will be used in the proof of the main theorems. In the fifth section, we prove the main theorems of the chapter.

5.2 Statement of main results

Let us consider the following assumption for an integer $m \geq 0$.

Assumption 5.2.1 (m)**.** *For $i,j \in \{0,\dots,d\}$, $a_t^{ij} = a_t^{ij}(x)$ are real-valued functions defined on $\Omega \times [0,T] \times \mathbf{R}^d$ that are $\mathcal{R}_T \otimes \mathcal{B}(\mathbf{R}^d)$-measurable and $\sigma_t^i = (\sigma_t^{i\varrho}(x))_{\varrho=1}^{\infty}$ are ℓ_2-valued functions that are $\mathcal{R}_T \otimes \mathcal{B}(\mathbf{R}^d)$-measurable. Moreover,*

(i) for each $(\omega,t) \in \Omega \times [0,T]$, the functions a_t^{ij} are $\max(m,1)$*-times continuously differentiable in x for all $i,j \in \{1,\dots,d\}$, a_t^{i0} and a_t^{0i} are m-time continuously differentiable in x for all $i \in \{0,1,\dots,d\}$, and σ_t^i are m-times continuously differentiable in x as ℓ_2-valued functions for all $i \in \{0,\dots,d\}$. Furthermore, there is a constant $K > 0$ such that for all $(\omega,t,x) \in \Omega \times [0,T] \times \mathbf{R}^d$,*

$$|\partial^\gamma a_t^{ij}| \leq K, \quad \forall\, i,j \in \{1,\dots,d\}, \quad \forall\, |\gamma| \leq \max(m,1),$$

$$|\partial^\gamma a_t^{i0}| + |\partial^\gamma a_t^{0i}| + |\partial^\gamma \sigma_t^i|_{\ell_2} \leq K, \quad \forall\, i \in \{0,\dots,d\}, \quad \forall |\gamma| \leq m;$$

(ii) there exists a positive constant $\varkappa > 0$ such that for all $(\omega,t,x) \in \Omega \times [0,T] \times \mathbf{R}^d$ and $\eta \in \mathbf{R}^d$

$$\sum_{i,j=1}^{d} \left(2a_t^{ij} - \sum_{\varrho=1}^{\infty} \sigma_t^{i\varrho}\sigma_t^{j\varrho}\right)\eta_i\eta_j \geq \varkappa|\eta|^2.$$

In this chapter, for each integer $m \geq 0$, we set $H^m = H^m(\mathbf{R}^d;\mathbf{R})$, $\mathbf{H}^m(\mathcal{F}_0)$, $\mathbf{H}^m = \mathbf{H}^m(\mathbf{R}^d;\mathbf{R})$, $\mathbf{H}^m(\ell_2) = \mathbf{H}^m(\mathbf{R}^d;\ell_2)$, $\mathbf{H}^m(\pi^2) = \mathbf{H}^\alpha(\mathbf{R}^d;\mathbf{R};\pi^2)$, and $\|\cdot\|_m = \|\cdot\|_{m,1}$, $(\cdot,\cdot)_m = (\cdot,\cdot)_{m,1}$, $\langle\cdot,\cdot\rangle_1 = \langle\cdot,\cdot\rangle_{1,1}$. We also set $C_c^\infty = C_c^\infty(\mathbf{R}^d;\mathbf{R}$.

Assumption 5.2.2 (m)**.** *We have $\phi \in \mathbf{H}^m(\mathcal{F}_0)$, $f \in \mathbf{H}^{m-1}$, $g \in \mathbf{H}^m(\ell_2)$, and $o \in \mathbf{H}^m(\pi^2)$. Set*

$$\kappa_m^2 = \mathbf{E}\left[\|\varphi\|_m^2\right] + \mathbf{E}\int_{]0,T]}\left(\|f_t\|_{m-1}^2 + \|g_t\|_{m,\ell_2}^2 + \int_{\mathbf{R}^d}\|o_t(z)\|_m^2\pi^2(dz)\right)dt.$$

For a real-valued twice continuous differentiable function ϕ on $\mathbf{R}^d$, it is easy to see that for all $x, z \in \mathbf{R}^d$,

$$\phi(x+z) - \phi(x) - \sum_{j=1}^{d} z^j \partial_j \phi(x) = \int_0^1 \sum_{i,j=1}^{d} z^i z^j \partial_{ij} \phi(x + \theta z)(1-\theta) d\theta. \tag{5.2.1}$$

For each $\delta \in (0, 1]$, let

$$\varsigma_1(\delta) = \int_{|z|\le\delta} |z|^2 \pi^1(dz), \quad \varsigma_2(\delta) = \int_{|z|\le\delta} |z|^2 \pi^2(dz), \quad \text{and} \quad \varsigma(\delta) = \varsigma_1(\delta) + \varsigma_2(\delta).$$

Fix $\delta \in (0, 1]$ such that

$$\varsigma(\delta) < \varkappa, \tag{5.2.2}$$

and notice that

$$\sum_{r=1}^{2} \pi_r(\{|z| > \delta\}) < \infty. \tag{5.2.3}$$

We write $I = I_\delta + I_{\delta^c}$, where

$$I_\delta \phi(x) := \int_{|z|\le\delta} \int_0^1 \sum_{i,j=1}^{d} z^i z^j \partial_{ij} \phi(x+\theta z)(1-\theta) d\theta \pi^1(dz)$$

and I_{δ^c} is defined as in (5.1.2) with integration over $\{|z| > \delta\}$ instead of $\mathbf{R}^d$.

Definition 5.2.1. An H^0-valued càdlàg adapted process u is called a solution of (5.1.1) if

(i) $u_t \in H^1$ for $d\mathbf{P} \times dt$-almost-every $(\omega, t) \in \Omega \times [0, T]$;

(ii) $\mathbf{E} \int_{]0,T]} \|u_t\|_1^2 dt < \infty$;

(iii) there exists a set $\tilde{\Omega} \subset \Omega$ of probability one such that for all $(\omega, t) \in [0, T] \times \tilde{\Omega}$ and $\phi \in C_c^\infty(\mathbf{R}^d)$,

$$(u_t, \phi)_0 = (\varphi, \phi)_0 + \int_{]0,t]} \left(\sum_{i,j=1}^{d} \left(\partial_j u_s, \partial_{-i}(a_s^{ij}\phi)\right)_0 + [\phi, f_s]_0 \right) ds$$

$$+ \int_{]0,t]} \int_{|z|\le\delta} \int_0^1 \sum_{i,j=1}^{d} \left(z^j \partial_j u_s(\cdot + \theta z), z^i \partial_{-i}\phi\right)_0 (1-\theta) d\theta \pi^1(dz) ds$$

$$+ \int_{]0,t]} \int_{|z|>\delta} \left(u_s(\cdot + z) - u_s - \mathbf{1}_{[-1,1]}(|z|) \sum_{j=1}^{d} z^j \partial_j u_s, \phi \right)_0 \pi^1(dz) ds$$

$$+\sum_{\varrho=1}^{\infty}\int_{]0,t]}\sum_{i=0}^{d}\left(\sigma_s^{i\varrho}\partial_i u_s+g_s^{\varrho},\phi\right)_0 dw_s^{\varrho}+\int_{]0,t]}\int_{\mathbf{R}^d}(u_{s-}(\cdot+z)-u_{s-}+o_t(z),\phi)_0\, q(dz,ds).$$

Remark 5.2.2. In the above definition, instead of δ we may choose any other positive constant.

The following existence theorem is a consequence of Theorems 2.9, 2.10, and 4.1 in [Gyö82] and will be verified in Section 4.

Theorem 5.2.3. *If Assumptions 5.2.1(m) and 5.2.2(m) hold with $m \geq 0$, then there exist a unique solution u of (5.1.1). Furthermore, u is a cádlág H^m-valued process with probability one and there is a constant $N = N(d,m,\varkappa,K,T)$ such that*

$$\mathbf{E}\left[\sup_{t\leq T}\|u_t\|_m^2\right]+\mathbf{E}\int_{]0,T]}\|u_s\|_{m+1}^2 ds\leq N\kappa_m^2. \tag{5.2.4}$$

Remark 5.2.4. We have used the standard definition of solution for the variational (or L^2) theory fo stochastic evolution equations. In what follows below, we will always assume $m \geq 2$, and so we have enough regularity to formulate the solution in the weak sense in (H^1, H^0, H^{-1}) without integrating by parts.

The following proposition is needed to establish the rate of convergence in time of our approximation scheme and is proved in Section 4.

Proposition 5.2.5. *Let Assumptions 5.2.1(m) and 5.2.2(m) hold for some $m \geq 1$ and u be the solution of (5.1.1). Moreover, assume that*

$$\sup_{t\leq T}\mathbf{E}\left[\|g_t\|_{m-1,\ell_2}^2\right]+\sup_{t\leq T}\mathbf{E}\int_{\mathbf{R}^d}\|o_t(z)\|_{m-1}^2\pi^2(dz)\leq K.$$

Then there is a constant $\lambda = \lambda(d,m,K,T,\varkappa,\kappa_m^2)$ such that for all $s,t\in[0,T]$,

$$\mathbf{E}\left[\|u_t-u_s\|_{m-1}^2\right]\leq\lambda|t-s|.$$

Assumption 5.2.3 (m)**.** *For $m \geq 3$, in addition to Assumption 5.2.2(m), there exists a random variable ξ with $E\xi < K$ such that for all $\omega\in\Omega$, $t,s\in[0,T]$,*

$$\|g_t\|_{m-1,\ell_2}^2+\int_{\mathbf{R}^d}\|o_t(z)\|_{m-1}^2\pi^2(dz)\leq\xi$$

$$\|f_t-f_s\|_{m-2}^2+\|g_t-g_s\|_{m-2,\ell_2}^2+\int_{\mathbf{R}^d}\|o_t(z)-o_s(z)\|_{m-1}^2\pi^2(dz)\leq\xi|t-s|.$$

Assumption 5.2.4 (m)**.** *For $m \geq 3$, in addition to Assumption 5.2.1 (i), there is a constant C such that for all $(\omega, x) \in \Omega \times \mathbf{R}^d$, $s, t \in [0, T]$, $i, j \in \{0, 1, \ldots, d\}$,*

$$|\partial^\gamma \left(a_t^{ij} - a_s^{ij}\right)|^2 + |\partial^\gamma \left(\sigma_t^i - \sigma_s^i\right)|_{\ell_2}^2 \leq C|t - s|, \quad \forall |\gamma| \leq m - 2.$$

We turn our attention to the discretisation of equation (5.1.1). For each $h \in \mathbf{R} - \{0\}$ and standard basis vector e_i, $i \in \{1, \ldots, d\}$, of $\mathbf{R}^d$ we define the first-order difference operator $\delta_{h,i}$ by

$$\delta_{h,i}\phi(x) := \frac{\phi(x + he_i) - \phi(x)}{h},$$

for all real-valued functions ϕ on $\mathbf{R}^d$. We define $\delta_{h,0}$ to be the identity operator. Notice that for all $\psi, \phi \in H^0$, we have

$$(\phi, \delta_{-h,i}\psi)_0 = -(\delta_{h,i}\phi, \psi)_0. \tag{5.2.5}$$

Set

$$\delta_i^h := \frac{1}{2}(\delta_{h,i} + \delta_{-h,i})$$

and observe that for all $\phi \in H^0$,

$$(\phi, \delta_i^h \phi)_0 = 0. \tag{5.2.6}$$

For each $h \neq 0$, we introduce the grid $\mathbf{G}_h := \{hz_k : z_k \in \mathbf{Z}^d, k \in \mathbf{N}_0, z_0 = 0\}$ with step size $|h|$. Let $\ell_2(\mathbf{G}_h)$ be the Hilbert space of real-valued functions ϕ on $\mathbf{G}_h$ such that

$$\|\phi\|_{\ell_2(\mathbf{G}_h)}^2 := |h|^d \sum_{x \in \mathbf{G}_h} |\phi(x)|^2 < \infty.$$

We approximate the operators $\mathcal{L}$ and $\mathcal{N}^\varrho$ by

$$\mathcal{L}_t^h \phi(x) := \sum_{i,j=0}^d a_t^{ij}(x)\delta_{h,i}\delta_{-h,j}\phi(x) \quad \text{and} \quad \mathcal{N}_t^{\varrho;h}\phi(x) := \sum_{i=0}^d \sigma_t^{i\varrho}(x)\delta_{h,i}\phi(x),$$

respectively. In order to approximate I, we approximate I_δ and I_{δ^c} separately. For each $k \in \mathbf{N} \cup \{0\}$ and $h \neq 0$, define the rectangles in $\mathbf{R}^d$

$$A_k^h := \left(z_k^1|h| - \frac{|h|}{2}, z_k^1|h| + \frac{|h|}{2}\right] \times \cdots \times \left(z_k^d|h| - \frac{|h|}{2}, z_k^d|h| + \frac{|h|}{2}\right],$$

where z_k^i, $i \in \{1, ..., d\}$, are the coordinates of $z_k \in \mathbf{Z}^d$, and set

$$B_k^h := A_k^h \cap \{|z| \leq \delta\}, \quad \bar{B}_k^h := A_k^h \cap \{|z| > \delta\}.$$

We approximate I_{δ^c} by

$$I^h_{\delta^c}\phi(x) := \sum_{k=0}^{\infty}\left((\phi(x+hz_k)-\phi(x))\,\bar{\zeta}_{h,k} - \sum_{i=1}^{d}\bar{\xi}^i_{h,k}\delta^h_i\phi(x)\right),$$

where

$$\bar{\zeta}_{h,k} := \pi^1(\bar{B}^h_k) \quad \text{and} \quad \bar{\xi}^i_{h,k} := \int_{\bar{B}^h_k\cap[-1,1]} z^i\pi^1(dz).$$

We continue with the approximation of the operator I_δ. By (5.2.1), for all $x\in\mathbf{G}_h$,

$$I_\delta\phi(x) = \sum_{k=0}^{\infty}\int_{B^h_k}\int_0^1\sum_{i,j=1}^{d} z^iz^j\partial_{ij}\phi(x+\theta z)(1-\theta)d\theta\pi^1(dz),$$

where there are only a finite number of non-zero terms in the infinite sum over k. The closest point in $\mathbf{G}_h$ to any point $z\in B^h_k$ is clearly hz_k. This simple observation leads us to the following (intermediate) approximation of $I_\delta\phi(x)$:

$$\sum_{k=0}^{\infty}\int_0^1\sum_{i,j=1}^{d}\int_{B^h_k} z^iz^j\pi^1(dz)\partial_{ij}\phi(x+\theta hz_k)(1-\theta)d\theta.$$

However, in order to ensure that our approximation is well-defined for functions $\phi\in\ell_2(\mathbf{G}_h)$, we need to approximate the integral over $\theta\in[0,1]$. Fix $k\in\mathbf{N}_0$ and $h\neq 0$. Consider the directed line segment $\{\theta hz_k : \theta\in[0,1]\}$ extending from the origin to the point $hz_k\in\mathbf{R}^d$. It is clear that this line segment intersects a unique finite sequence of rectangles from the set $\{A^h_{\bar{k}}\}_{\bar{k}\in\mathbf{N}_0}$. Denote the number of rectangles by $\chi(h,k)$. Since the line's start point is the origin, the first rectangle it intersects is A^h_0, and since the line's endpoint is hz_k, the last rectangle it intersects is A^h_k, the center of which is the point hz_k. If $\chi(h,k)>2$, then in between these two rectangles, the line segment intersects $\chi(h,k)-2$ additional rectangles from the set $\{A^h_{\bar{k}}\}_{\bar{k}\in\mathbf{N}_0}-\{A^h_0\cup A^h_k\}$. Denote the indices of these rectangles by $r^{h,k}_l$, $l\in\{2,\dots,\chi(h,k)-1\}$, and set $r^{h,k}_1=0$ and $r^{h,k}_{\chi(h,k)}=k$; that is, $\{\theta hz_k;\theta\in[0,1]\}\subseteq\cup_{l=1}^{\chi(h,k)}A^h_{r^{h,k}_l}$. Corresponding to the set of rectangles $\{A^h_{r^{h,k}_l}\}_{l=1}^{\chi(h,k)}$ is a partition $0=\theta^{h,k}_0\le\cdots\le\theta^{h,k}_{\chi(h,k)}=1$ of the interval $[0,1]$ such that for each $l\in\{1,\dots,\chi(h,k)\}$ and $\theta\in(\theta^{h,k}_{l-1},\theta^{h,k}_l)$, $\theta hz_k\in A^h_{r^{h,k}_l}$. Since the diagonal of a d-dimensional hypercube with side length $|h|$ has length $\sqrt{d}h$, for each $k\in\mathbf{N}_0$, $z\in B^h_k$, and $l\in\{1,\dots,\chi(h,k)\}$,

$$|\theta z - hz_{r^{h,k}_l}| \le |\theta z-\theta hz_k| + |\theta hz_k - hz_{r^{h,k}_l}| \le \sqrt{d}|h|, \tag{5.2.7}$$

for all $\theta \in (\theta_{l-1}^{h,k}, \theta_l^{h,k})$. Set

$$\zeta_{h,k}^{ij} = \int_{B_k^h} z^i z^j \pi^1(dz), \quad \bar{\theta}_l^{h,k} = \int_{\theta_{l-1}^{h,k}}^{\theta_l^{h,k}} (1-\theta)d\theta$$

and define the operator

$$I_\delta^h \phi(x) =: \sum_{k=0}^{\infty} \sum_{l=1}^{\chi(h,k)} \bar{\theta}_l^{h,k} \sum_{i,j=1}^{d} \zeta_{h,k}^{ij} \delta_{h,i}\delta_{-h,j}\phi(x + hz_{r_l^{h,k}}),$$

where there are only a finite number of non-zero terms in the infinite sum over k. Set $I^h = I_\delta^h + I_{\delta^c}^h$ and introduce the martingales

$$p_t^{h,k,i} = \int_{]0,t]} \int_{B_k^h} z^i q(dt, dz), \quad \bar{p}_t^{h,k} = q(\bar{B}_k^h,]0,t]).$$

Moreover, set

$$\tilde{\theta}_l^{h,k} := \theta_{l+1}^{h,k} - \theta_l^{h,k}.$$

Let $\mathcal{T} \geq 1$ be an integer and set $\tau = T/\mathcal{T}$ and $t_n = n\tau$ for $i \in \{0, 1, \dots, \mathcal{T}\}$. For any $\mathbf{F}$-martingale $(p_t)_{t\leq T}$, we use the notation $\Delta p_{n+1} := p_{t_{n+1}} - p_{t_n}$. Define recursively the $\ell_2(\mathbf{G}_h)$-valued random variables $(\hat{u}_n^{h,\tau})_{n=0}^{\mathcal{T}}$ by

$$\begin{aligned}
\hat{u}_n^{h,\tau}(x) =& \hat{u}_{n-1}^{h,\tau}(x) + \left((\mathcal{L}_{t_{n-1}}^h + I^h)\hat{u}_{n-1}^{h,\tau}(x) + f_{t_{n-1}}(x)\right)\tau + \sum_{\varrho=1}^{\infty} (\mathcal{N}_{t_{n-1}}^{\varrho;h}\hat{u}_{n-1}^{h,\tau}(x) + g_{t_{n-1}}^{\varrho}(x))\Delta w_n^{\varrho} \\
&+ \sum_{k=0}^{\infty}\sum_{i=1}^{d}\left(\sum_{l=1}^{\chi(h,k)} \tilde{\theta}_l^{h,k}\delta_{h,i}\hat{u}_{n-1}^{h,\tau}(x + hz_{r_l^{h,k}})\right)\Delta p_n^{h,k,i} + \int_{\mathbf{R}^d} o_{t_{n-1}}(x,z)q(]t_{n-1}, t_n], dz) \\
&+ \sum_{k=0}^{\infty}\left(\hat{u}_{n-1}^{h,\tau}(x + hz_k) - \hat{u}_{n-1}^{h,\tau}(x)\right)\Delta \bar{p}_n^{h,k}, \quad n \in \{1, \dots, \mathcal{T}\},
\end{aligned} \tag{5.2.8}$$

with initial condition

$$\hat{u}_0^{h,\tau}(x) = \varphi(x), \quad x \in \mathbf{G}_h$$

It is clear that $\hat{u}_n^{h,\tau}$ is $\mathcal{F}_{t_n}$-measurable for every $n \in \{0, 1, \dots, \mathcal{T}\}$. Define the operators

$$\tilde{\mathcal{L}}_t^h \phi = \sum_{i,j=0}^{d} a_t^{ij}\delta_{h,i}\delta_{-h,j}\phi - \pi^1(\{|z| > \delta\})\phi - \sum_{i=1}^{d}\int_{\delta<|z|\leq 1} z^i\pi^1(dz)\delta_i^h\phi$$

and

$$\tilde{I}_{\delta^c}^h \phi = \sum_{k=0}^{\infty} \phi(x + hz_k)\overline{\zeta}_{h,k}$$

and note that $\tilde{\mathcal{L}}^h + \tilde{I}_{\delta^c}^h + I_\delta = \mathcal{L}^h + I^h$. On $\mathbf{G}_h$, we also consider the following implicit-explicit discretization scheme of (5.1.1):

$$
\begin{aligned}
\hat{v}_n^{h,\tau}(x) =& \hat{v}_{n-1}^{h,\tau}(x) + \Big((\tilde{\mathcal{L}}_{t_n}^h + I_\delta^h)\hat{v}_n^{h,\tau}(x) + \tilde{I}_{\delta^c}^h v_{n-1}^{h,\tau}(x) + f_{t_n}(x) \Big)\tau \\
&+ \mathbf{1}_{n>1}\sum_{\varrho=1}^{\infty}(\mathcal{N}_{t_{n-1}}^{\varrho;h}\hat{v}_{n-1}^{h,\tau}(x) + g_{t_{n-1}}^{\varrho}(x))\Delta w_n^{\varrho} \\
&+ \mathbf{1}_{n>1}\sum_{k=0}^{\infty}\sum_{i=1}^{d}\left(\sum_{l=1}^{\chi(h,k)}\tilde{\theta}_l^{h,k}\delta_{h,i}\hat{v}_{n-1}^{h,\tau}(x + hz_{r_l^{h,k}})\right)\Delta p_n^{h,k,i} + \int_{\mathbf{R}^d} o_{t_{n-1}}(x,z)q(]t_{n-1},t_n],dz) \\
&+ \mathbf{1}_{n>1}\sum_{k=0}^{\infty}\Big(\hat{v}_{n-1}^{h,\tau}(x+hz_k) - \hat{v}_{n-1}^{h,\tau}(x)\Big)\Delta \bar{p}_n^{h,k}, \quad n \in \{1,\dots,\mathcal{T}\},
\end{aligned}
\tag{5.2.9}
$$

with initial condition

$$\hat{v}_0^{h,\tau}(x) = \varphi(x), \quad x \in \mathbf{G}_h,$$

where $\mathbf{1}_{n>1} = 0$ if $n = 1$ and $\mathbf{1}_{n>1} = 1$ if $n \geq 2$. A solution $(\hat{v}_n^{h,\tau})_{n=0}^M$ of (5.2.9) is understood as a sequence of $\ell_2(\mathbf{G}_h)$-valued random variables such that $\hat{v}_n^{h,\tau}$ is $\mathcal{F}_{t_n}$-measurable for every $n \in \{0,1,\dots,M\}$ and satisfies (5.1.1).

Remark 5.2.6. Under Assumptions 5.2.2 and 5.2.3, for $m > 2 + d/2$, by virtue of the embedding $H^{m-2} \hookrightarrow \ell_2(\mathbf{G}^h)$, the free-terms f, g, and $o(z)$ are continuous $\ell_2(\mathbf{G}^h)$ valued processes, and consequently the above schemes make sense. Moreover, for $0 < |h| < 1$, there is a constant N independent of h such that -

$$\|\phi\|_{\ell_2(\mathbf{G}^h)} \leq N\|\phi\|_{m-2}. \tag{5.2.10}$$

Assumption 5.2.5. *The parameters $h \neq 0$ and $\mathcal{T}$ are such that*

$$d\frac{\tau}{h^2} < \frac{\varkappa - \varsigma(\delta)}{\left(2\left(\sup_{t,x,\omega}\sum_{i,j=1}^d |a_t^{ij}(x)|^2\right)^{1/2} + \varsigma_1(\delta)\right)^2}. \tag{5.2.11}$$

Remark 5.2.7. We have assumed that the coefficients a_t^{ij} were bounded uniformly in ω, t, and x, so that quantity in in denominator (5.2.11) is well-defined.

The following are our main theorems.

Theorem 5.2.8. *Let Assumptions 5.2.1(m) through 5.2.4(m) hold for some $m > 2 + \frac{d}{2}$ and let Assumption 5.2.5 hold. Let u be the solution of (5.1.1) and let $(\hat{u}_n^{h,\tau})_{n=0}^{\mathcal{T}}$ be defined by (5.2.8). Then there is a constant $N = N(d,m,\varkappa,K,T,C,\lambda,\kappa_m^2,\delta)$ such that for any real number h with $0 < |h| < 1$,*

$$\mathbf{E}\left[\max_{0\leq n\leq\mathcal{T}}\sup_{x\in\mathbf{G}_h}|u_{t_n}(x) - \hat{u}_n^{h,\tau}(x)|^2\right] + \mathbf{E}\left[\max_{0\leq n\leq\mathcal{T}}\|u_{t_n} - \hat{u}_n^{h,\tau}\|_{\ell_2(\mathbf{G}_h)}^2\right] \leq N\Big(|h|^2 + \tau\Big).$$

Theorem 5.2.9. *Let Assumptions 5.2.1(m) through 5.2.4(m) hold for some $m > 2 + \frac{d}{2}$ and let u be a solution of (5.1.1). There exists a constant $R = R(d, m, \varkappa, K, \delta)$ such that if $\mathcal{T} > R$, then there exists a unique solution $(\hat{v}_n^{h,\tau})_{n=0}^{\mathcal{T}}$ of (5.2.9) and a constant $N = N(d, m, \varkappa, K, T, C, \lambda, \kappa_m^2, \delta)$ such that for any real number h with $0 < |h| < 1$,*

$$\mathbf{E}\left[\max_{0\le n\le\mathcal{T}}\sup_{x\in\mathbf{G}_h}|u_{t_n}(x) - \hat{v}_n^{h,\tau}(x)|^2\right] + \mathbf{E}\left[\max_{0\le n\le\mathcal{T}}\|u_{t_n} - \hat{v}_n^{h,\tau}\|^2_{\ell_2(\mathbf{G}_h)}\right] \le N\left(|h|^2 + \tau\right).$$

Remark 5.2.10. In fact, with more regularity, as consequence of Corollaries 5.5.3 and 5.5.4, we can obtain stronger error estimates with difference operators in the error norms. This is an immediate consequence of the approach we use to obtaining these estimates.

5.3 Simulation

Let us consider finite difference approximations for the following SIDE on $[0, T] \times \mathbf{R}^d$:

$$\begin{aligned} du_t(x) &= \left(\left(\frac{\bar{\sigma}_1^2}{2} + \frac{\bar{\sigma}_2^2}{2}\right)\partial_1^2 u_t(x) + \int_{\mathbf{R}} (u_t(x+z) - u_t(x) - \partial_1 u_t(x) z)\pi(dz)\right) dt + \bar{\sigma}_2 \partial_1 u_t(x) dw_t \\ &\quad + \int_{\mathbf{R}} (u(x+z) - u(x))\, q(dt, dz), \\ u_0(x) &= \frac{1}{\sqrt{2\pi}\bar{\sigma}_0} \exp\left(-\frac{x^2}{\bar{\sigma}_1^2\bar{\sigma}_0^2}\right), \end{aligned} \tag{5.3.1}$$

where $\pi(dz) = c_- \exp(-\beta_- z) \frac{dz}{|z|^{1+\alpha_-}} \mathbf{1}_{(-\infty,0)}(z) + c_+ \exp(-\beta_+ z) \frac{dz}{|z|^{1+\alpha_+}} \mathbf{1}_{(0,\infty)}(z)$. It is easily verified that for $(t, x) \in [0, T] \times \mathbf{R}^d$,

$$v_t(x) = \frac{1}{\sqrt{\pi(2\bar{\sigma}_0^2 + 4t)}} \exp\left(\frac{x^2}{\bar{\sigma}_1^2(\bar{\sigma}_0^2 + 2t)}\right)$$

solves

$$dv_t(x) = \frac{\bar{\sigma}_1^2}{2}\partial_1^2 v_t(x) dt, \quad v_0(x) = \frac{1}{\sqrt{2\pi}\bar{\sigma}_0} \exp\left(-\frac{x^2}{\bar{\sigma}_1^2\bar{\sigma}_0^2}\right).$$

Moreover, applying Itô's formula, we find that

$$u_t(x) = v_t\left(x + \bar{\sigma}_2 w_t + \int_{\mathbf{R}^d} z q(dt, dz)\right) \tag{5.3.2}$$

solves (5.3.1). Thus, we can compare our finite difference approximations with (5.3.2).

In our numerical simulations, we used MATLAB 2013a and made the following parameter specification:

$$\bar{\sigma}_1 = \frac{1}{2}, \quad \bar{\sigma}_2 = \frac{1}{4}, \quad \bar{\sigma}_0 = \frac{1}{2}, \quad c_- = c_+ = 1, \quad \beta_- = \beta_+ = 1, \quad \alpha_- = \alpha_+ = 1.1, \quad T = 1.$$

We also made a few practical simplifications. Both the explicit and implicit-explicit approximations were assumed to take the value zero on $(-\infty, 8] \cup [8, \infty)$. We also restricted the support of $\pi(dz)$ to $[-3, 3]$. We would like to investigate the associated error with these reductions in the future. Regarding the first reduction, we mention that a good heuristic is to choose the size of domain according to the support of the free terms and the exit time of the diffusion for which the drift of the SIDE (up to zero order terms) is the infinitesimal generator of. In fact, it is more than a heuristic and we aim to address this in a future work.

In our simulation, we took $\delta = \frac{1}{100}$. It follows that $\kappa = \bar{\sigma}_1^2 = \frac{1}{2}$ and

$$\begin{aligned} \varsigma(\delta) &= c_- \int_0^\delta \exp(-\beta_- z) z^{1-\alpha_-} dz + c_+ \int_0^\delta \exp(-\beta_+ z) z^{1-\alpha_+} dz + z \\ &= c_- \beta_-^{\alpha_- - 2} \gamma(2 - \alpha_-, \beta_- \delta) + c_+ \beta_+^{\alpha_+ - 2} \gamma(2 - \alpha_+, \beta_+ \delta) \approx 0.0082, \end{aligned}$$

where $\gamma(\eta, z)$ denotes the lower incomplete gamma function. Thus, the right-hand-side of (5.2.11) is approximately 1.0559, and hence we can always set $\tau = h^2$. The quantities $\zeta_{h,k}^{11}, \bar{\zeta}_{h,k}$, and $\xi_{h,k}^1$ can all be calculated using MATLAB's built-in upper and lower incomplete gamma functions, or by implementing an appropriate numerical integration procedure. The calculation of $\theta_l^{h,k}, \bar{\theta}_l^{h,k}$, and $\tilde{\theta}_l^{h,k}$ are all straightforward in one-dimension. Some more thought would need to spent on how to calculate these quantities in higher dimensions. Of course as an alternative, one could set $\delta = \frac{h}{2}$, but then the schemes are not guaranteed to converge as h tends to zero. This is the drawback of taking $\delta = \frac{h}{2}$ and not including the additional terms in I_δ (see the paragraph at the bottom of page 1620 in [CV05]). It does seem that the method we propose to discretise I_δ is novel in this respect. In our error analysis, we have considered $h \in \{2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}, 2^{-7}\}$ and $\tau = h^2$.

The term

$$\int_{|z|>\delta} (u_t(x+z) - u_t(x))\, \pi(dz)$$

in the drift of (5.3.1) can be cancelled with the compensator of the compensated Poisson random measure term. We get a similar cancellation in the corresponding finite difference equations, and thus we can replace $\bar{p}_t^{h,k} = q(\bar{B}_k^h,]t_n, t_{n+1}])$ with $\hat{p}_t^{h,k} = p(\bar{B}_k^h,]t_n, t_{n+1}])$ in the explicit 5.2.8 and implicit-explicit (5.2.9) scheme.

In order to simulate

$$\Delta p_n^{h,k} = \int_{]t_n, t_{n+1}]} \int_{B_k^h} z q(dt, dz), \quad \hat{p}_t^{h,k} = p(\bar{B}_k^h,]t_n, t_{n+1}]),$$

for the finest time step size $\tau = 2^{-14}$, we used the algorithm discussed in Section 4 of [KM11]. In this algorithm, a parameter ϵ is chosen for which the process $\Delta p_n^{h,0} = \int_{]0,t]}\int_{|z|<\epsilon} zq(dt,dz)$ is approximated by a Wiener process with infinitesimal variance $\int_{|z|<\epsilon} z^2\pi(dz)$. We chose the parameter $\epsilon = 2^{-8}$, which is one-half times the smallest step size h under consideration in our error analysis. The process $\int_{]0,t]}\int_{|z|>\epsilon} zq(dt,dz) = \int_{]0,t]}\int_{|z|>\epsilon} zp(dt,dz)$ (we have used symmetry of the measure $\pi(dz)$)) is a compound Poisson process with jump intensity

$$\lambda := 2\int_{\epsilon}^{3} \pi(dz) \approx 68.9676$$

and jump-size density

$$\bar{f}(z) = \frac{1}{\lambda}\left(c_- \exp\left(-\beta_- z\right) \frac{dz}{|z|^{1+\alpha_-}} \mathbf{1}_{(-3,2^{-8})}(z) + c_+ \exp\left(-\beta_+ z\right) \frac{dz}{|z|^{1+\alpha_+}} \mathbf{1}_{(2^{-8},3)}(z)\right).$$

The underlying Poisson process was simulated using MATLAB's built-in Poisson random variable generator; of course there are other simple methods that one can use as an alternative (e.g. exponential times or uniform times for fixed number of jumps). We sampled random variables from the density $\bar{f}$ by sampling the positive and negative parts separately and using an acceptance-rejection algorithm with a Pareto random variable. We refer to [KM11] for more details. Once we simulated the point process on $[0,T]\times[-3,-\epsilon]\cup[\epsilon,3]$, we then computed $\int_{]0,t]}\int_{|z|>\epsilon} zp(dt,dz)$. In order to compute $\hat{p}_t^{h,k} = p(\bar{B}_k^h,]t_n, t_{n+1}])$, we ran a histogram with the intervals $\bar{B}_k^h$.

The quantity $\Delta p_n^{h,k} = \int_{]t_n,t_{n+1}]}\int_{B_k^h} zq(dt,dz)$ is zero for $k \neq 0$ when $h < \frac{\delta}{2}$ (for $h \in \{2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}\}$) since $B_k^h = \emptyset$ for $k \neq 0$ when $h < \frac{\delta}{2}$. For $h \in \{2^{-6}, 2^{-7}\}$, $\Delta p_n^{h,k}$ is non-zero for $k \in \{-1, 0, 1\}$. A similar analysis holds for the quantity $\zeta_{h,k}^{11}$. As mentioned above, we set $\hat{p}_t^{h,0}$ equal to the Weiner process approximating the small jumps. To compute $\int_{]t_n,t_{n+1}]}\int_{B_k^h} zq(dt,dz)$ for $k \in \{-1,1\}$ in the case $h \in \{2^{-6}, 2^{-7}\}$, we summed the jump sizes in their respective bins and compensated. To obtain the above quantities for coarser time step sizes, we cumulatively summed the finer increments and took the union of jump sizes.

Lastly, we made use of the Fast Fourier Transform to compute terms of the form

$$\sum_{k=0}^{\infty} \phi(x + hz_k)\Delta\hat{p}_n^{h,k},$$

which would be quite computationally expensive otherwise. In our error analysis, we ran 3000 simulations of the explicit and implicit-explicit schemes on 30 CPUs and computed the following quantities:

By our main theorems and the relation $\tau = h^2$, these errors should proportional to h (i.e. $O(h)$). This is precisely what we observe in Figure 5.1. The slight bump down at the finest two spatial step-sizes $h \in \{2^{-6}, 2^{-7}\}$ is most likely due to the increase in the number

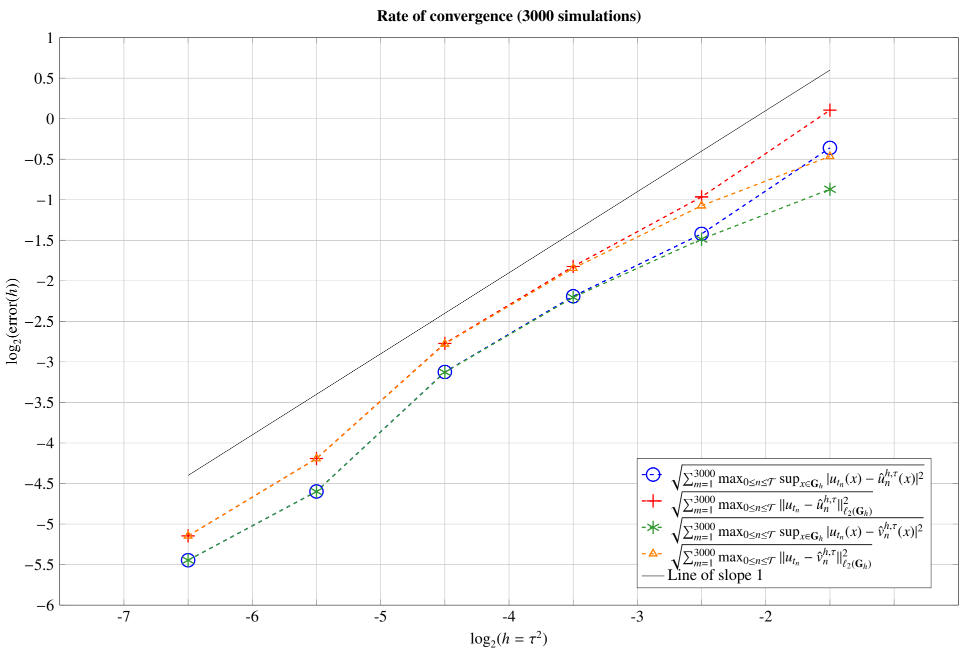

Figure 5.1 Simulated errors with respect to the space discretization and a line as reference slope on a $\log_2$ scale

.

of terms in the approximation of I^h_δ (three to be precise) and the analogous small jump term in the noise.

5.4 Auxiliary results

In this section, we present some results that will be needed for the proof of Theorems 5.2.8 and 5.2.9. Introduce the operators

$$
\begin{aligned}
\mathcal{I}^{\delta;h}(z)\phi(x) &:= \sum_{k=0}^{\infty} \mathbf{1}_{B_k^h}(z) \sum_{l=1}^{\chi(h,k)} \sum_{i=1}^{d} \tilde{\theta}_l^{h,k} z^i \delta_{h,i}\phi(x + hz_{r_l^{h,k}}),\\
\mathcal{I}^{\delta^c;h}(z)\phi(x) &:= \sum_{k=0}^{\infty} \mathbf{1}_{\bar{B}_k^h}(z)(\phi(x + hz_k) - \phi(x)),\\
\mathcal{I}^h(z)\phi(x) &:= \mathcal{I}^{\delta;h}(z)\phi(x) + \mathcal{I}^{\delta^c;h}(z)\phi(x).
\end{aligned}
$$

Consider the following explicit and implicit-explicit schemes in H^0:

$$
\begin{aligned}
u_n^{h,\tau} =& u_{n-1}^{h,\tau} + \left((\mathcal{L}^h_{t_{n-1}} + I^h)u_{n-1}^{h,\tau} + f_{t_{n-1}}\right)\tau + \sum_{\varrho=1}^{\infty}(\mathcal{N}^{\varrho;h}_{t_{n-1}} u_{n-1}^{h,\tau} + g^{\varrho}_{t_{n-1}})\Delta w_n^{\varrho}\\
&+ \int_{\mathbf{R}^d} \left(\mathcal{I}^h(z)u_{n-1}^{h,\tau} + o_{t_{n-1}}(z)\right) q(dz,]t_{n-1}, t_n]), \quad n \in \{1, \dots, \mathcal{T}\}, \qquad (5.4.1)
\end{aligned}
$$

and

$$
\begin{aligned}
v_n^{h,\tau} =& v_{n-1}^{h,\tau} + \Big((\tilde{\mathcal{L}}_{t_n}^h + I_\delta^h)v_n^{h,\tau} + \tilde{I}_{\delta^c}^h v_{n-1}^{h,\tau} + f_{t_n}\Big)\tau + \mathbf{1}_{n>1}\sum_{\varrho=1}^{\infty}(\mathcal{N}_{t_{n-1}}^{\varrho;h} v_{n-1}^{h,\tau} + g_{t_{n-1}}^{\varrho})\Delta w_n^{\varrho} \\
&+ \mathbf{1}_{n>1}\int_{\mathbf{R}^d}\Big(\mathcal{I}^h(z)v_{n-1}^{h,\tau} + o_{t_{n-1}}(z)\Big)q(dz,]t_{n-1}, t_n]), \quad n \in \{1, \ldots, \mathcal{T}\}, \qquad (5.4.2)
\end{aligned}
$$

with initial condition

$$u_0^{h,\tau}(x) = v_0^{h,\tau}(x) = \varphi(x), \quad x \in \mathbf{R}^d.$$

We now prove some lemmas that will help us to establish the consistency of our approximations. The following lemma is well-known and we omit the proof (see, e.g., [GK10]).

Lemma 5.4.1. *For each integer $m \geq 0$, there is a constant $N = N(d,m)$ such that for all $u \in H^{m+2}$ and $v \in H^{m+3}$,*

$$
\begin{aligned}
\|\delta_{h,i}u - \partial_i u\|_m &\leq \frac{1}{2}|h|\|u\|_{m+2}, \\
\|\delta_{h,i}\delta_{-h,j}v - \partial_{ij}v\|_m &\leq N|h|\|v\|_{m+3}.
\end{aligned}
$$

Lemma 5.4.2. *For each integer $m \geq 0$, there is a constant $N = N(d,m,\delta)$ such that for all $u \in H^{m+3}$, we have*

$$\|Iu - I^h u\|_m \leq N|h|\|u\|_{m+3}. \qquad (5.4.3)$$

Proof. It suffices to show (5.4.3) for $u \in C_c^\infty(\mathbf{R}^d)$. We begin with $m = 0$. A simple calculation shows that

$$
\begin{aligned}
I_{\delta^c}u(x) - I_{\delta^c}^h u(x) =& \sum_{k=0}^{\infty}\int_{\bar{B}_k^h}\int_0^1\sum_{i=1}^{d}(z^i - hz_k^i)\partial_i u(x + hz_k + \theta(z - hz_k))d\theta\pi^1(dz) \\
&- \sum_{k=0}^{\infty}\int_{\bar{B}_k^h\cap[-1,1]}\sum_{i=1}^{d} z^i(\partial_i u(x) - \delta_i^h u(x))\pi^1(dz).
\end{aligned}
$$

By Minkowski's inequality, we get

$$
\begin{aligned}
\|I_{\delta^c}u - I_{\delta^c}^h u\|_0 \leq& \sum_{k=0}^{\infty}\int_{\bar{B}_k^h}\sum_{i=1}^{d}|z^i - hz_k^i|\|\partial_i u\|_0\pi^1(dz) \\
+\sum_{k=0}^{\infty}\int_{\bar{B}_k^h\cap[-1,1]}\sum_{i=1}^{d}|z^i|\|\partial_i u(x) - \delta_i^h u(x)\|_0\pi^1(dz) &\leq N|h|\|u\|_3 + N\sum_{i=1}^{d}\|\partial_i u(x) - \delta_i^h u(x)\|_0,
\end{aligned}
$$

since $|z - hz_k| \leq |h|\sqrt{d}/2$ and (5.2.3) holds. Thus, by Lemma 5.4.1, we have

$$\|I_{\delta^c}u - I_{\delta^c}^h u\|_0 \leq N|h|\|u\|_3. \qquad (5.4.4)$$

We also have

$$I_\delta u(x) - I^h_\delta u(x) =$$

$$\sum_{k=0}^{\infty}\int_{B^h_k}\sum_{l=1}^{\chi(h,k)}\int_{\theta^{h,k}_{l-1}}^{\theta^{h,k}_l}\sum_{i,j=1}^{d} z^i z^j\left(\partial_{ij}u(x+\theta z)-\delta_{h,i}\delta_{-h,j}u(x+hz_{r^{h,k}_l})\right)(1-\theta)d\theta\pi^1(dz). \tag{5.4.5}$$

Note that

$$\begin{aligned}
&\partial_{ij}u(x+\theta z)-\delta_{h,i}\delta_{-h,j}u(x+hz_{r^{h,k}_l})\\
&=\partial_{ij}u(x+\theta z)-\partial_{ij}u(x+hz_{r^{h,k}_l})+\partial_{ij}u(x+hz_{r^{h,k}_l})-\delta_{h,i}\delta_{-h,j}u(x+hz_{r^{h,k}_l})\\
&=\int_0^1\sum_{q=1}^{d}\left(\theta z^q-hz^q_{r^{h,k}_l}\right)\partial_q\partial_{ij}u\left(x+hz_{r^{h,k}_l}+\rho(\theta z-hz_{r^{h,k}_l})\right)d\rho\\
&\quad+\partial_{ij}u(x+hz_{r^{h,k}_l})-\delta_{h,i}\delta_{-h,j}u(x+hz_{r^{h,k}_l}).
\end{aligned}$$

By (5.2.7), we have $|\theta z^q - hz^q_{r^{h,k}_l}| \leq N|h|$. Hence, substituting the above relation in (5.4.5), using Minkowski's inequality, (5.2.2), and Lemma 5.4.1, we obtain

$$||I_\delta u - I^h_\delta u||_0 \leq |h|N||u||_3. \tag{5.4.6}$$

Combining (5.4.4) and (5.4.6), we have (5.4.3) for $m=0$. The case $m>0$ follows from the case $m=0$, since for a multi-index γ, we have

$$\partial^\gamma(Iu - I^h u) = I\partial^\gamma u - I^h\partial^\gamma u.$$

□

Lemma 5.4.3. *For each integer $m \geq 0$, there is a constant $N = N(d,m,\delta)$, there is a constant such that for all $u \in H^{m+2}$, we have*

$$\int_{\mathbf{R}^d}||\mathcal{I}^h(z)u - \mathcal{I}(z)u||^2_m\pi^2(dz) \leq N|h|^2||u||^2_{m+2}. \tag{5.4.7}$$

Proof. It suffices to prove the lemma for $u \in C^\infty_c(\mathbf{R}^d)$ and $m=0$. We have

$$\mathcal{I}^\delta(z)u(x) - \mathcal{I}^{\delta;h}(z)u(x) =$$

$$\sum_{k=0}^{\infty}\mathbf{1}_{B^h_k}(z)\sum_{l=1}^{\chi(h,k)}\int_{\theta^{h,k}_{l-1}}^{\theta^{h,k}_l}\sum_{i=1}^{d} z^i(\partial_i u(x+\theta z)-\delta_{h,i}u(x+hz_{r^{h,k}_l}))d\theta.$$

Notice that

$$\partial_i u(x+\theta z) - \delta_{h,i}u(x+hz_{r_l^{h,k}}) = \int_0^1 \sum_{i,j=1}^d \partial_{ij}u(x+\rho(\theta z - hz_{r_l^{h,k}}))(\theta z^j - hz^j_{r_l^{h,k}})d\rho$$
$$+\partial_i u(x+hz_{r_l^{h,k}}) - \delta_{h,i}u(x+hz_{r_l^{h,k}}).$$

Thus, by Remark 5.2.7 and Lemma 5.4.1, we get

$$\|\mathcal{I}^{\delta;h}(z)u - \mathcal{I}^{\delta}(z)u\|_0^2 \leq \mathbf{1}_{|z|\leq\delta}|z|^2N|h|^2\|u\|_2^2,$$

and hence by (5.2.2), we obtain

$$\int_{\mathbf{R}^d} \|\mathcal{I}^{\delta;h}(z)u - \mathcal{I}^{\delta}(z)u\|_0^2\pi^2(dz) \leq N|h|^2\|u\|_2^2. \tag{5.4.8}$$

We also have

$$|\mathcal{I}^{\delta^c}(z)u(x) - \mathcal{I}^{\delta^c;h}(z)u(x)| = \sum_{k=0}^{\infty} \mathbf{1}_{\bar{B}_k^h}(z)|u(x+z) - u(x+hz_k)|$$
$$\leq \sum_{k=0}^{\infty} \mathbf{1}_{\bar{B}_k^h}(z) \int_0^1 \sum_{i=1}^d |\partial_i u(x+hz_k+\rho(z-hz_k))||z^i - hz_k^i|d\rho.$$

Consequently,

$$\|\mathcal{I}^{\delta^c;h}(z)u - \mathcal{I}^{\delta^c}(z)u\|_0^2 \leq \mathbf{1}_{|z|>\delta}N|h|^2\|u\|_1^2,$$

which implies by (5.2.3) that

$$\int_{\mathbf{R}^d} \|\mathcal{I}^{\delta^c;h}(z)u - \mathcal{I}^{\delta^c}(z)u\|_0^2\pi^2(dz) \leq N|h|^2\|u\|_1^2. \tag{5.4.9}$$

Combining (5.4.9) and (5.4.8), we have (5.4.7) for $m = 0$. The case $m > 0$ follows from the case $m = 0$, since for a multi-index γ, we have

$$\partial^\gamma(\mathcal{I}u - \mathcal{I}^h u) = \mathcal{I}\partial^\gamma u - \mathcal{I}^h\partial^\gamma u.$$

□

Lemma 5.4.4. *If Assumption 5.2.1(m) holds for some $m \geq 0$, then for any $\epsilon \in (0,1)$ there exists constants $N_1 = N_1(d,m,\varkappa,K,\delta,\epsilon)$ and $N_2 = N_2(d,m,\varkappa,K,\delta,\epsilon)$ such that for any $u \in H^m$,*

$$\mathbf{G}_t^{(m)}(u) := 2(u,\mathcal{L}_t^h u)_m + \|\mathcal{N}_t^h u\|_{m,\ell_2}^2 + 2(u,I^h u)_m + \int_{\mathbf{R}^d} \|\mathcal{I}^h(z)u\|_m^2\pi^2(dz)$$

$$\leq -(\varkappa - \varsigma(\delta) - \epsilon) \sum_{i=1}^{d} ||\delta_{h,i}u||_m^2 + N_1 ||u||_m^2, \tag{5.4.10}$$

and

$$(u, \tilde{\mathcal{L}}_t^h u)_m + (u, I_\delta^h u)_m \leq -(\varkappa - \varsigma_1(\delta) - \epsilon) \sum_{i=1}^{d} ||\delta_{h,i}u||_m^2 + N_2 ||u||_n^m. \tag{5.4.11}$$

Proof. By virtue of Lemma 3.1 and Theorem 3.2 in [GK10], under Assumption 5.2.1, there is a constant $N = N(d, m, \varkappa)$ such that for any $u \in H^m$ and $\epsilon > 0$,

$$2(u, \mathcal{L}_t^h u)_m + ||\mathcal{N}_t^h u||_{m,\ell_2}^2 \leq -(\varkappa - \epsilon) \sum_{i=1}^{d} ||\delta_{h,i}u||_m^2 + N||u||_m^2.$$

Therefore, it suffices to show that there is a constant $N = N(\delta)$ such that for all $u \in C_c^\infty(\mathbf{R}^d)$,

$$2(u, I^h u)_m + \int_{\mathbf{R}^d} ||\mathcal{I}^h(z)u||_m^2 \pi^2(dz) \leq \varsigma(\delta) \sum_{i=1}^{d} ||\delta_{h,i}u||_m^2 + N||u||_m^2. \tag{5.4.12}$$

We start with $m = 0$. Since

$$(u, I_\delta^h u)_0 = \sum_{k=0}^{\infty} \int_{B_k^h} \sum_{l=1}^{\chi(h,k)} \sum_{i,j=1}^{d} \bar{\theta}_l^{k,h} z^i z^j \int_{\mathbf{R}^d} \delta_{h,i}\delta_{-h,j} u(x + hz_{r_l^{h,k}}) u(x) dx \pi^1(dz)$$

and

$$\int_{\mathbf{R}^d} \delta_{h,i}\delta_{-h,j} u(x + hz_{r_l^{h,k}}) u(x) dx = -\int_{\mathbf{R}^d} \delta_{h,i} u(x + hz_{r_l^{h,k}}) \delta_{h,j} u(x) dx,$$

by Hölder's inequality, we get

$$2(u, I_\delta^h u)_0 \leq \int_{|z| \leq \delta} |z|^2 \pi^1(dz) \sum_{i=1}^{d} ||\delta_{h,i}u||_0^2 = \varsigma_1(\delta) \sum_{i=1}^{d} ||\delta_{h,i}u||_0^2.$$

In addition, owing to Holder's inequality and (5.2.6), we have

$$2(u, I_{\delta^c}^h u)_0 = \sum_{k=0}^{\infty} \int_{\bar{B}_k^h} \int_{\mathbf{R}^d} \left(u(x + hz_k) - u(x) - \mathbf{1}_{[-1,1]}(z) \sum_{i=1}^{d} z^i \delta_i^h u(x) \right) u(x) dx \pi^1(dz) \leq 0.$$

By Minkowski's inequality, we have

$$||\mathcal{I}^{\delta;h}(z)u||^2 \leq \sum_{k=0}^{\infty} \mathbf{1}_{B_k^h}(z)|z|^2 \sum_{i=1}^{d} ||\delta_{h,i}u||_0^2 \ \text{ and } \ ||\mathcal{I}^{\delta^c;h}(z)u||_0^2 \leq 4 \sum_{k=0}^{\infty} \mathbf{1}_{\bar{B}_k^h}(z) ||u||_0^2$$

and hence

$$\int_{\mathbf{R}^d} ||\mathcal{I}^h(z)u||_0^2 \pi^2(dz) \leq \varsigma_2(\delta) \sum_{i=1}^{d} ||\delta_{h,i}u||_0^2 + 4\pi^1(\{|z| > \delta\}) ||u||_0^2,$$

which proves (5.4.12) for $m = 0$. The case $m > 0$ follows by replacing u with $\partial^\gamma u$ for $|\gamma| \leq m$. This proves (5.4.10), which implies (5.4.11). □

Remark 5.4.5. It follows that for $m \geq 0$, there is a constant $N_5 = N_5(d, m, K, \delta)$ such that for any $u \in H^m$,

$$\|\mathcal{N}_t^{\varrho;h} u\|_m^2 + \int_{\mathbf{R}^d} \|\mathcal{I}^h(z)u\|_m^2 \pi^2(dz) \leq N_5 \sum_{i=0}^d \|\delta_{h,i} u\|_m^2 \tag{5.4.13}$$

$$\leq N_5 \left(1 + \frac{4d}{h^2}\right) \|u\|_m^2. \tag{5.4.14}$$

Lemma 5.4.6. *For any $m \geq 0$ and $u \in H^m$,*

$$\|\tilde{I}_{\delta^c}^h u\|_m^2 \leq \pi^1(\{|z| > \delta\})^2 \|u\|_m^2. \tag{5.4.15}$$

Moreover, if Assumption 5.2.1 holds for some $m \geq 0$, then for any $\epsilon > 0$ and $u \in H^m$,

$$\|(\mathcal{L}_t^h + I^h)u\|_m^2 \leq (1+\varepsilon)\frac{N_3 d}{h^2} \sum_{i=1}^d \|\delta_{h,i} u\|_m^2 + N_4 \left(1 + \frac{1}{h^2}\right) \|u\|_m^2 \tag{5.4.16}$$

where

$$N_3 := \left(2\left(\sup_{t,x,\omega} \sum_{i,j=1}^d |a^{ij}(x)|^2\right)^{1/2} + \varsigma_1(\delta)\right)^2$$

and N_4 is a constant depending only on d, m, K, δ, and ϵ.

Proof. It suffices to prove the lemma for $u \in C_c^\infty(\mathbf{R}^d)$. It follows that

$$(\mathcal{L}_t^h + I_\delta^h)u(x) = \sum_{k=0}^\infty \sum_{l=1}^{\chi(h,k)} \bar{\theta}_l^{h,k} \sum_{i,j=1}^d \hat{\zeta}_{t,h,k}^{ij}(x) \delta_{h,i}\delta_{-h,j} u(x + hz_{r_l^{h,k}}) + \sum_{\substack{i,j=0 \\ i \text{ or } j=0}}^d a_t^{ij} \delta_{h,i}\delta_{-h,j} u(x)$$

where $\hat{\zeta}_{t,h,k}^{ij}(x) := \zeta_{h,k}^{ij}$ for $k \neq 0$ and $\hat{\zeta}_{t,h,0}^{ij}(x) := \zeta_{h,0}^{ij} + 2a_t^{ij}(x)$ (recall that $\bar{\theta}_1^{h,0} = \frac{1}{2}$ and $\chi(h, 0) = 1$). Moreover, for each multi-index γ with $1 \leq |\gamma| \leq m$,

$$\begin{aligned}
\partial^\gamma(\mathcal{L}_t^h + I_\delta^h)u(x) &= \sum_{k=0}^\infty \sum_{l=1}^{\chi(h,k)} \bar{\theta}_l^{h,k} \sum_{i,j=1}^d \hat{\zeta}_{h,k}^{ij}(x) \delta_{h,i}\delta_{-h,j} \partial^\gamma u(x + hz_{r_l^{h,k}}) \\
&\quad + \sum_{\{\beta : \beta < \gamma\}} N(\beta, \gamma) \sum_{i,j=1}^d \left(\partial^{\gamma-\beta} a_t^{ij}(x)\right) \delta_{h,i}\delta_{-h,j} \partial^\beta u(x) \\
&\quad + \sum_{\{\beta : \beta \leq \gamma\}} N(\beta, \gamma) \sum_{\substack{i,j=0 \\ i \text{ or } j=0}}^d \left(\left(\partial^{\gamma-\beta} a_t^{ij}(x)\right) \delta_{h,i}\delta_{-h,j} \partial^\beta u(x)\right)
\end{aligned}$$

$$=: (A_1(\gamma) + A_2(\gamma) + A_3(\gamma))u(x),$$

where $N(\beta,\gamma)$ are constants depending only on β and γ. By Young's inequality and Jensen's inequality, for any $\epsilon \in (0,1)$, we have

$$\|(\mathcal{L}_t^h + I^h)u\|_m^2 \leq (1+\epsilon)\sum_{|\gamma|\leq m}\|A_1(\gamma)u\|_0^2$$
$$+3\left(1+\frac{1}{\epsilon}\right)\left[\sum_{|\gamma|\leq m}\left(\|A_2(\gamma)u\|_0^2 + \|A_3(\gamma)u\|_0^2\right) + \|I_{\delta^c}^h u\|_m^2\right].$$

Applying Minkowski's inequality and the Cauchy-Bunyakovsky-Schwarz inequality and noting that $\sum_{l=1}^{\chi(h,k)}\bar{\theta}_l^{h,k} = \frac{1}{2}$ and

$$\|\delta_{h,i}\partial^\beta u\|_0 \leq \frac{2}{h}\|\partial^\beta u\|_0;\ \forall i\in\{0,1\ldots,d\},\quad \forall|\beta| = m,$$

we obtain

$$\|A_1(\gamma)u\|_0 \leq \sum_{k=0}^{\infty}\sum_{l=1}^{\chi(h,k)}\bar{\theta}_l^{h,k}\left(\sup_{t,x,\omega}\sum_{i,j=1}^d\left|\hat{\zeta}_{h,k}^{ij}(x)\right|^2\right)^{1/2}\left(\sum_{i,j=1}^d\left\|\delta_{h,i}\delta_{-h,j}u(\cdot + hz_{r_l^{h,k}})\right\|_m^2\right)^{1/2}$$
$$\leq \frac{\sqrt{d}}{h}\sum_{k=0}^{\infty}\left(\sup_{t,x,\omega}\sum_{i,j=1}^d|\hat{\zeta}_{h,k}^{ij}(x)|^2\right)^{1/2}\left(\sum_{i=1}^d\|\delta_{h,i}\partial^\gamma u\|_0^2\right)^{1/2}$$

and

$$\sum_{k=0}^{\infty}\left(\sup_{t,x,\omega}\sum_{i,j=1}^d|\hat{\zeta}_{h,k}^{ij}(x)|^2\right)^{1/2} = \left(\sup_{t,x,\omega}\sum_{i,j=1}^d\left|\int_{B_0^h}z^iz^j\pi^1(dz) + 2a_t^{ij}(x)\right|^2\right)^{1/2}$$
$$+\sum_{k=1}^{\infty}\left(\sum_{i,j=1}^d\left|\int_{B_k^h}z^iz^j\pi^1(dz)\right|^2\right)^{1/2} \leq 2\left(\sup_{t,x,\omega}\sum_{i,j=1}^d|a_t^{ij}(x)|^2\right)^{1/2} + \varsigma(\delta).$$

Thus,

$$\sum_{|\gamma|\leq m}\|A_1(\gamma)u\|_0^2 \leq \frac{N_3 d}{h^2}\sum_{i=1}^d\|\partial_{h,i}u\|_m^2.$$

Another application of the Cauchy-Bunyakovsky-Schwarz inequality and Minkowski's inequality, combined with the inequalities

$$\|\delta_{h,i}\partial^\beta u\|_0 \leq \|\partial_i\partial^\beta u\|_0\quad \forall i\in\{0,1\ldots,d\},\quad \forall|\beta|\leq m-1,$$
$$\|\delta_{h,i}\delta_{-h,j}\partial^\beta u\|_0 \leq \|\partial_{ij}\partial^\beta u\|_0,\ \forall; i,j\in\{1,\ldots,d\},\quad \forall|\beta|\leq m-2,$$

and

$$\|\delta_{h,i}\delta_{-h,j}\partial^{\beta}u\|_0 \le \frac{2}{h}\|\delta_{h,i}u\|_m, \quad \forall; i,j \in \{1,\ldots,d\}, \quad \forall|\beta| = m-1,$$

yields

$$\sum_{|\gamma|\le m}\left(\|A_2(\gamma)u\|_0^2 + \|A_3(\gamma)\|_0^2\right) \le N\left(1+\frac{1}{h^2}\right)\|u\|_m^2.$$

By Minkowski's integral inequality, we have

$$\|I^h_{\delta^c}u\|_m \le \int_{\mathbf{R}^d}\sum_{k=0}^{\infty}\mathbf{1}_{\overline{B}^h_k}\|u(\cdot+hz_k) - u - \mathbf{1}_{[-1,1]}(z)\sum_{i=1}^{d} z^i\delta_{h,i}u\|_m\pi^1(dz)$$

$$\le 3\left(\pi^1(\{|z|>\delta\}) + \frac{2d\int_{\delta<|z|\le 1}|z|\pi^1(dz)}{h}\right)\|u\|_m.$$

It is also easy to see that (5.4.15) holds. Combining above inequalities, we obtain (5.4.16). □

The following theorem establishes the stability of the explicit approximate scheme (5.4.1).

Theorem 5.4.7. *Let Assumption 5.2.1 hold with $m \ge 0$ and Assumption 5.2.5 hold. Let $F^i \in \mathbf{H}^m$ for $i \in \{0,...,d\}$, $G \in \mathbf{H}^m(\ell_2)$, and $R \in \mathbf{H}^m(\pi^2)$. Consider the following scheme in H^m:*

$$u_n^{h,\tau} = u_{n-1}^{h,\tau} + \int_{]t_{n-1},t_n]}\left((\mathcal{L}^h_{t_{n-1}} + I^h)u_{n-1}^{h,\tau} + \sum_{i=0}^{d}\delta_{h,i}F_t^i\right)dt + \int_{]t_{n-1},t_n]}\left(N^{\varrho;h}_{t_{n-1}}u_{n-1}^{h,\tau} + G_t^{\varrho}\right)dw_t^{\varrho}$$

$$+ \int_{]t_{n-1},t_n]}\int_{\mathbf{R}^d}\left(\mathcal{I}^h(z)u_{n-1}^{h,\tau} + R_t(z)\right)q(dt,dz), \quad n \in \{1,\ldots,\mathcal{T}\},$$

for any H^m−valued $\mathscr{F}_0$−measurable initial condition φ. If $\phi \in \mathbf{H}^m(\mathcal{F}_0)$, then there is a constant $N = N(d,m,\varkappa,K,T,\delta)$ such that

$$\mathbf{E}\left[\max_{0\le n\le\mathcal{T}}\|u_n^{h,\tau}\|_m^2\right] + \mathbf{E}\sum_{n=0}^{\mathcal{T}}\tau\sum_{i=0}^{d}\|\delta_{h,i}u_n^{h,\tau}\|_m^2 \le N\mathbf{E}\left[\|\varphi\|_m^2\right]$$

$$+N\mathbf{E}\int_0^T\Big(\sum_{i=0}^{d}\|F_t^i\|_m^2 + \|G_t\|_m^2 + \int_{\mathbf{R}^d}\|R_t(z)\|_m^2\pi^2(dz)\Big)dt. \tag{5.4.17}$$

Proof. If $\mathbf{E}\left[\|\varphi\|_m^2\right] < \infty$, then proceeding by induction on n and using Young's and Jensen's inequality, Itô's isometry, (5.4.16), and (5.4.14), we get that for all $n \in \{0,1,\ldots,\mathcal{T}\}$, $\mathbf{E}\left[\|u_n^{h,\tau}\|_m^2\right] < \infty$. Applying the identity $\|y\|_m^2 - \|x\|_m^2 = 2(x, y-x)_m + \|y-x\|_m^2$, $x,y \in H^m$, for each $n \in \{1, \dots, \mathcal{T}\}$, we obtain

$$\|u_n^{h,\tau}\|_m^2 = \|u_{n-1}^{h,\tau}\|_m^2 + \sum_{i=1}^{6} I_i(t_n), \tag{5.4.18}$$

where

$$\begin{aligned}
I_1(t_n) &:= 2\tau(u_{n-1}^{h,\tau}, \left(\mathcal{L}_{t_{n-1}}^h + I^h\right) u_{n-1}^{h,\tau})_m + \|\eta(t_n)\|_m^2, \\
I_2(t_n) &:= 2\int_{]t_{n-1},t_n]} \sum_{i=0}^{d} (u_{n-1}^{h,\tau}, \delta_{h,i} F_t^i)_m dt, \\
I_3(t_n) &:= \left\| \tau \left(\mathcal{L}_{t_{n-1}}^h + I^h\right) u_{n-1}^{h,\tau} + \int_{]t_{n-1},t_n]} \sum_{i=0}^{d} \delta_{h,i} F_t^i dt \right\|_m^2, \\
I_4(t_n) &:= 2\int_{]t_{n-1},t_n]} \left(u_{n-1}^{h,\tau}, \mathcal{N}_{t_{n-1}}^{\varrho;h} u_{n-1}^{h,\tau} + G_t^{\varrho}\right)_m dw_t^{\varrho}, \\
I_5(t_n) &:= 2\int_{]t_{n-1},t_n]} \int_{\mathbf{R}^d} \left(u_{n-1}^{h,\tau}, \mathcal{I}^h(z) u_{n-1}^{h,\tau} + R_t(z)\right)_m q(dt,dz), \\
I_6(t_n) &:= 2\left(\tau(\mathcal{L}_{t_{n-1}}^h + I^h) u_{n-1}^{h,\tau}, \eta(t_n)\right)_m + 2\left(\int_{]t_{n-1},t_n]} \sum_{i=0}^{d} \delta_{h,i} F_t^i dt, \eta(t_n)\right)_m,
\end{aligned}$$

and where

$$\eta(t_n) := \int_{]t_{n-1},t_n]} \left(\mathcal{N}_{t_{n-1}}^{\varrho;h} u_{n-1}^{h,\tau} + G_t^{\varrho}\right) dw_t^{\varrho} + \int_{]t_{n-1},t_n]} \int_{\mathbf{R}^d} \left(\mathcal{I}^h(z) u_{n-1}^{h,\tau} + R_t(z)\right) q(dt,dz).$$

By virtue of Assumption 5.2.5, we fix $\tilde{q} > 0$ and $\epsilon > 0$ small enough such that

$$\overline{q} := \varkappa - \varsigma(\delta) - \epsilon - (1+\epsilon)(1+\tilde{q}) N_3 d \frac{\tau}{h^2} - \tilde{q} > 0,$$

where N_3 is the constant in (5.4.6). Since the two stochastic integrals that define η are orthogonal square-integrable martingales, by Young's inequality and (5.4.13), for all $\mathfrak{q} > 0$,

$$\begin{aligned}
\mathbf{E}\left[\|\eta(t_n)\|_m^2\right] \le \mathbf{E}\left[\tau \|\mathcal{N}_{t_{n-1}}^h u_{n-1}^{h,\tau}\|_{m,\ell_2}^2\right] + \mathbf{E}\tau \int_{\mathbf{R}^d} \|\mathcal{I}^h(z) u_{n-1}^{h,\tau}\|_m^2 \pi^2(dz) + \mathfrak{q}\mathbf{E}\tau \sum_{i=0}^{d} \|\delta_{h,i} u_{n-1}^{h,\tau}\|_m^2 \\
+ \left(1 + \frac{N_5}{\mathfrak{q}}\right) \mathbf{E} \int_{]t_{n-1},t_n]} \left(\|G_t\|_{m,\ell_2}^2 + \int_{\mathbf{R}^d} \|R_t(z)\|_m^2 \pi^2(dz)\right) dt.
\end{aligned} \tag{5.4.19}$$

Thus, taking $\mathfrak{q} = \frac{\tilde{q}}{3}$ in (5.4.19), we have

$$E I_1(t_n) \le \mathbf{E}\tau \mathbf{G}_{t_{l-1}}^{(m)}(u_{l-1}^{h,\tau}) + \frac{\tilde{q}}{3} \mathbf{E}\tau \sum_{i=0}^{d} \|\delta_{h,i} u_{n-1}^{h,\tau}\|_m^2$$

$$+\left(1+\frac{3N_5}{\tilde{q}}\right)\mathbf{E}\int_{]t_{n-1},t_n]}\left(\|G_t\|_{m,\ell_2}^2+\int_{\mathbf{R}^d}\|R_t(z)\|_m^2\pi^2(dz)\right)dt.$$

Using (5.2.5) and Young's inequality, we obtain

$$EI_2(t_n)\le\frac{\tilde{q}}{3}\mathbf{E}\tau\sum_{i=0}^{d}\|\delta_{h,i}u_{n-1}^{h,\tau}\|_m^2+\frac{3}{\tilde{q}}\mathbf{E}\int_{]t_{n-1},t_n]}\sum_{i=0}^{d}\|F_t^i\|_m^2dt.$$

An application of Young's inequality and (5.4.16) yields

$$EI_3(t_n)\le(1+\epsilon)(1+\tilde{q})N_3d\frac{\tau}{h^2}\mathbf{E}\tau\sum_{i=1}^{d}\|\delta_{h,i}u_{n-1}^{h,,\tau}\|_m^2+(1+\tilde{q})N_4\left(\tau+\frac{\tau}{h^2}\right)\mathbf{E}\left[\tau\|u_{n-1}^{h,\tau}\|_m^2\right]$$
$$+(d+1)\left(1+\frac{1}{\tilde{q}}\right)\mathbf{E}\int_{]t_{n-1},t_n]}\left(\tau\|F_t^0\|_m^2+\frac{4d\tau}{h^2}\sum_{i=1}^{d}\|F_t^i\|_m^2\right)dt.$$

Making use of the estimate (5.4.14) and noting that $\mathbf{E}\|u_n^{h,\tau}\|_m^2<\infty$, $G\in\mathbf{H}^m(\ell_2)$, and $R\in\mathbf{H}^m(\pi^2)$, we obtain $EI_4(t_n)=EI_5(t_n)=0$. Moreover, as $(\mathcal{L}_{t_{n-1}}^h+I^h)u_{n-1}^{h,\tau}$ is $\mathcal{F}_{t_{n-1}}$-measurable and $E(\eta(t_n)|\mathcal{F}_{t_{n-1}})=0$, the expectation of first term in $I_6(t_n)$ is zero, and hence by Young's inequality, for any $\mathfrak{q}_1>0$,

$$EI_6(t_n)\le\mathfrak{q}_1\mathbf{E}\left[\|\eta(t_n)\|_m^2\right]+\frac{1}{\mathfrak{q}_1}E\left\|\int_{]t_{n-1},t_n]}\sum_{i=0}^{d}\delta_{h,i}F_t^idt\right\|_m^2.$$

Moreover, by Jensen's inequality, (5.4.19), and (5.4.13), for any $\mathfrak{q}_1>0$ and $\mathfrak{q}>0$,

$$EI_6(t_n)\le(\mathfrak{q}_1\mathfrak{q}+\mathfrak{q}_1N_5)\mathbf{E}\tau\sum_{i=0}^{d}\|\delta_{h,i}u_{n-1}^{h,\tau}\|_m^2$$
$$+\mathbf{E}\int_{]t_{n-1},t_n]}\left(\frac{(d+1)\tau}{\mathfrak{q}_1}\|F_t^0\|_m^2+\frac{4d(d+1)\tau}{\mathfrak{q}_1h^2}\sum_{i=1}^{d}\|F_t^i\|_m^2\right)dt$$
$$+\mathfrak{q}_1\left(1+\frac{N_5}{\mathfrak{q}}\right)\mathbf{E}\int_{]t_{n-1},t_n]}\left(\|G_t\|_{m,\ell_2}^2+\int_{\mathbf{R}^d}\|R_t(z)\|_m^2\pi^2(dz)\right)dt.$$

We choose $\mathfrak{q}$ and $\mathfrak{q}_1$ such that $\mathfrak{q}_1\mathfrak{q}+\mathfrak{q}_1N_5\le\tilde{q}/3$. Thus, owing to (5.4.10), we have

$$E\mathbf{G}_{t_{n-1}}^{(m)}(u_{n-1}^{h,\tau})+\left(\tilde{q}+(1+\epsilon)(1+\tilde{q})N_3d\frac{\tau}{h^2}\right)\mathbf{E}\tau\sum_{i=1}^{d}\|\delta_{h,i}u_{n-1}^{h,\tau}\|_m^2$$

$$\le-\overline{q}\mathbf{E}\tau\sum_{i=1}^{d}\|\delta_{h,i}u_{n-1}^{h,\tau}\|_m^2+N_1\mathbf{E}\left[\tau\|u_{n-1}^{h,\tau}\|_m^2\right].$$

Taking the expectation of both sides of (5.4.18), summing-up, and combining the above inequalities and identities, we find that there is a constant $N = N(d, m, \varkappa, K, \delta)$ such that for all $n \in \{0, 1, \ldots, \mathcal{T}\}$,

$$\begin{aligned}\mathbf{E}\left[\|u_n^{h,\tau}\|_m^2\right] \le \mathbf{E}\left[\|\varphi\|_m^2\right] - \overline{q}\mathbf{E}\sum_{l=1}^{n}\tau\sum_{i=1}^{d}\|\delta_{h,i}u_{l-1}^{h,\tau}\|_m^2 \\ +\left(N_1 + \tilde{q} + (1+\tilde{q})N_4\left(\tau + \frac{\tau}{h^2}\right)\right)\mathbf{E}\sum_{l=1}^{n}\tau\|u_{l-1}^{h,\tau}\|_m^2 \\ +N\left(\tau + \frac{\tau}{h^2}\right)\mathbf{E}\int_{]0,t_n]}\sum_{i=0}^{d}\|F_t^i\|_m^2 dt + N\mathbf{E}\int_{]0,t_n]}\left(\|G_t\|_{m,\ell_2}^2 + \int_{\mathbf{R}^d}\|R_t(z)\|_m^2\pi^2(dz)\right)dt.\end{aligned}$$

Therefore, by discrete Gronwall's inequality, there is a constant $N = N(d, m, \varkappa, K, T, \delta)$ such that

$$\begin{aligned}\mathbf{E}\left[\|u_n^{h,\tau}\|_m^2\right] + \mathbf{E}\sum_{l=0}^{n}\tau\sum_{i=0}^{d}\|\delta_{h,i}u_l^{h,\tau}\|_m^2 \le N\mathbf{E}\left[\|\varphi\|_m^2\right] \\ +N\mathbf{E}\int_{]0,T]}\left(\sum_{i=0}^{d}\|F_t^i\|_m^2 + \|G_t\|_{m,\ell_2}^2 + \int_{\mathbf{R}^d}\|R_t(z)\|_m^2\pi^2(dz)\right)dt.\end{aligned} \tag{5.4.20}$$

Now that we have proved (5.4.20), we will show (5.4.17). Estimating as we did above, we get that there is a constant N such that

$$\begin{aligned}\mathbf{E}\max_{0\le n\le\mathcal{T}}\sum_{l=1}^{n}(I_1(t_l) + I_2(t_l) + I_3(t_l) + I_6(t_l)) \le N\mathbf{E}\sum_{l=0}^{\mathcal{T}-1}\tau\sum_{i=0}^{d}\|\delta_{h,i}u_l^{h,\tau}\|_m^2 \\ +N\mathbf{E}\int_{]0,T]}\left(\sum_{i=0}^{d}\|F_t^i\|_m^2 + \|G_t\|_{m,\ell_2}^2 + \int_{\mathbf{R}^d}\|R_t(z)\|_m^2\pi^2(dz)\right)dt.\end{aligned}$$

Applying the Burkholder-Davis-Gundy inequality and Young's inequality, we obtain

$$\begin{aligned}\mathbf{E}\max_{0\le n\le\mathcal{T}}\sum_{l=1}^{n}I_5(t_l) \le 6\mathbf{E}\left|\sum_{l=1}^{n}\int_{]t_{n-1},t_n]}\int_{\mathbf{R}^d}\left(u_{n-1}^{h,\tau}, \mathcal{I}^h(z)u_{n-1}^{h,\tau} + R_t(z)\right)_m^2\pi^2(dz)dt\right|^{1/2} \\ \le \frac{1}{4}\mathbf{E}\left[\max_{0\le n\le\mathcal{T}}\|u_n^{h,\tau}\|_m^2\right] + N\left(\mathbf{E}\sum_{l=0}^{\mathcal{T}-1}\tau\mathbf{E}\|\delta_{h,i}u_l^{h,\tau}\|_m^2 + \mathbf{E}\sum_{l=0}^{\mathcal{T}-1}\tau\mathbf{E}\|u_l^{h,\tau}\|_m^2\right) \\ +N\mathbf{E}\int_{]0,T]}\int_{\mathbf{R}^d}\|R_t(z)\|_m^2\pi^2(dz)dt.\end{aligned}$$

We can estimate $\mathbf{E}\max_{0\le n\le\mathcal{T}}\sum_{l=1}^{n}I_4(t_l)$ in similar way. Combining the above $\mathbf{E}\max_{0\le n\le\mathcal{T}}$-estimates and (5.4.20), we obtain (5.4.17). $\square$

The following theorem establishes the existence and uniqueness of a solution to (5.4.2) and the stability of the implicit-explicit approximation scheme.

Theorem 5.4.8. *Let Assumption 5.2.1 hold with $m \geq 0$. Let $F^i \in \mathbf{H}^m$ for $i \in \{0,...,d\}$, $G \in \mathbf{H}^m(\ell_2)$ and $R \in \mathbf{H}^m(\pi^2)$. Then there exists a constant $R = R(d, m, \varkappa, K, \delta)$ such that if $\mathcal{T} > R$, then for any $h \neq 0$, there exists a unique H^m-valued solution $(v_n^{h,\tau})_{n=0}^{\mathcal{T}}$ of*

$$
\begin{aligned}
v_n^{h,\tau} = v_{n-1}^{h,\tau} &+ \int_{]t_{n-1},t_n]} \left((\tilde{\mathcal{L}}_{t_n}^h + I_\delta^h) v_n^{h,\tau} + \tilde{I}_{\delta^c}^h v_{n-1}^{h,\tau} + \sum_{i=0}^d \delta_{h,i} F_t^i \right) dt \\
&+ \int_{]t_{n-1},t_n]} \left(\mathbf{1}_{n>1} \mathcal{N}_{t_{n-1}}^{\varrho;h} v_{n-1}^{h,\tau} + G_t^\varrho \right) dw_t^\varrho \\
&+ \int_{]t_{n-1},t_n]} \int_{\mathbf{R}^d} \left(\mathbf{1}_{n>1} \mathcal{I}^h(z) v_{n-1}^{h,\tau} + R_t(z) \right) q(dt, dz),
\end{aligned} \tag{5.4.21}
$$

for $n \in \{1, \ldots, \mathcal{T}\}$, for any H^m–valued $\mathscr{F}_0$–measurable initial condition φ. Moreover, if $\phi \in \mathbf{H}^m(\mathcal{F}_0)$, then there is a constant $N = N(d, m, \varkappa, K, T, \delta)$ such that

$$
\mathbf{E}\left[\max_{0 \leq n \leq \mathcal{T}} \|v_n^{h,\tau}\|_m^2\right] + \mathbf{E} \sum_{n=0}^{\mathcal{T}} \tau \sum_{i=0}^d \|\delta_{h,i} v_n^{h,\tau}\|_m^2 \leq N\mathbf{E}\left[\|\varphi\|_m^2\right]
$$

$$
+N\mathbf{E} \int_0^T \left(\sum_{i=0}^d \|F_t^i\|_m^2 + \|G_t\|_m^2 + \int_{\mathbf{R}^d} \|R_t(z)\|_m^2 \pi^2(dz) \right) dt. \tag{5.4.22}
$$

Proof. For each $n \in \{1, \ldots, \mathcal{T}\}$, we write (5.4.21) as

$$
D_n v_n^{h,\tau} = y_{n-1},
$$

where D_n is the operator defined by

$$
D_n \phi := \phi - \tau \left(\tilde{\mathcal{L}}_{t_n}^h + I_\delta^h \right) \phi
$$

and

$$
\begin{aligned}
y_{n-1} := v_{n-1}^{h,\tau} &+ \int_{]t_{n-1},t_n]} \left(\tilde{I}_{\delta^c}^h v_{n-1}^{h,\tau} + \sum_{i=0}^d \delta_{h,i} F_t^i \right) dt + \int_{]t_{n-1},t_n]} \left(\mathbf{1}_{n>1} \mathcal{N}_{t_{n-1}}^{\varrho;h} v_{n-1}^{h,\tau} + G_t^\varrho \right) dw_t^\varrho \\
&+ \int_{]t_{n-1},t_n]} \int_{\mathbf{R}^d} \left((\mathbf{1}_{n>1} \mathcal{I}^h(z) v_{n-1}^{h,\tau} + R_t(z) \right) q(dt, dz).
\end{aligned}
$$

Fix ϵ_1 and ϵ_2 in $(0, 1)$ such that

$$
\overline{q}_1 := \varkappa - \varsigma_1(\delta) - \epsilon_1 > 0.
$$

and

$$
\overline{q}_2 := \varkappa - \varsigma(\delta) - \epsilon_2 > 0.
$$

Owing to Lemma 5.4.6, there is a constant $N = N(d, m, K, \delta)$ such that for all $\phi \in H^m$,

$$\|D_n\phi\|_m^2 \leq N\left(1 + \tau^2\left(\frac{1}{h^2} + \frac{1}{h^4}\right)\right)\|\phi\|_m^2. \tag{5.4.23}$$

Assume $\mathcal{T} > TN_2$. By (5.4.11), for all $\phi \in H^m$, we have

$$(\phi, D_n\phi)_m \geq (1 - \tau N_2)\|\phi\|_m^2 + \overline{q}_1\tau\sum_{i=1}^{d}\|\delta_{h,i}\phi\|_m^2 \geq (1 - \tau N_2)\|\phi\|_m^2. \tag{5.4.24}$$

Using Jensen's inequality and (5.4.15), we get

$$\begin{aligned}\|y_0\|_m^2 \leq 5\left(1 + \pi^1(\{|z| > \delta\})^2\tau^2\right)\|\phi\|_m^2 + \frac{20\tau}{h^2}\int_{]0,t_1]}\sum_{i=0}^{d}\|F_t^i\|_m^2 dt + 5\left\|\int_{]0,t_1]} G_t^{\varrho} dw_t^{\varrho}\right\|_m^2 \\ + 5\left\|\int_{]0,t_1]}\int_{\mathbf{R}^d} R_t(z)q(dt, dz)\right\|_m^2.\end{aligned} \tag{5.4.25}$$

Since $\varphi \in H^m$, $F^i \in \mathbf{H}^m$, $i \in \{0, 1, \ldots, d\}$, $G \in \mathbf{H}^m(\ell_2)$, and $R \in \mathbf{H}^m(\pi^2)$, it follows that $y_0 \in H^m$. By (5.4.23), and (5.4.24), owing to Proposition 3.4 in [GM05] ($p = 2$), there exists a unique $v_1^{h,\tau}$ in H^m such that $D_1v_1^{h,\tau} = y_0$, and moreover

$$\|v_1^{h,\tau}\|_m^2 \leq 1 + \frac{\|y_0\|_m^2}{(1 - \tau N_2)^2} < \infty. \tag{5.4.26}$$

Proceeding by induction on $n \in \{1, \ldots, \mathcal{T}\}$, one can show that there exists a unique $v_n^{h,\tau}$ in H^m such that $D_nv_n^{h,\tau} = y_{n-1}$, and moreover

$$\|v_n^{h,\tau}\|_m^2 \leq 1 + \frac{\|y_{n-1}\|_m^2}{(1 - \tau N_2)^2} < \infty. \tag{5.4.27}$$

Assume that $\mathbf{E}\|\varphi\|_m^2 < \infty$. By (5.4.25) and (5.4.26) and the fact that $f^i \in \mathbf{H}^m$, $i \in \{0, 1, \ldots, d\}$, $g \in \mathbf{H}^m(\ell_2)$, and $R \in \mathbf{H}^m(\pi^2)$, it follows that $\mathbf{E}\|v_1^{h,\tau}\|_m^2 < \infty$. By Jensen's inequality, (5.4.15), and (5.4.14), we have

$$\begin{aligned}\mathbf{E}\left[\|y_{n-1}\|_m^2\right] \leq 7N\left(1 + \pi^1(\{|z| > \delta\})^2\tau^2 + \mathbf{1}_{n>1}\tau\left(1 + \frac{1}{h^2}\right)\right)\mathbf{E}\left[\|v_{n-1}^{h,\tau}\|_m^2\right] \\ + \frac{28\tau}{h^2}\mathbf{E}\int_{]0,t_1]}\sum_{i=0}^{d}\|F_t^i\|_m^2 dt + 7\mathbf{E}\int_{]0,t_1]}\|G_t\|_{m,\ell_2}^2 dt + 7\mathbf{E}\int_{]0,t_1]}\int_{\mathbf{R}^d}\|R_t(z)\|_m^2\pi^2(dz)dt.\end{aligned} \tag{5.4.28}$$

Proceeding by induction on n and combining (5.4.27) and (5.4.28), we obtain

$$\mathbf{E}\left[\|v_n^{h,\tau}\|_m^2\right] < \infty, \quad \forall n \in \{0, 1, \ldots, \mathcal{T}\}. \tag{5.4.29}$$

Applying the identity $\|y\|_m^2-\|x\|_m^2 = 2(x,y-x)_m+\|y-x\|_m^2$, $x,y\in H^m$, for any $n\in\{1,\dots,\mathcal{T}\}$, we have

$$\|v_n^{h,\tau}\|_m^2 = \|v_{n-1}^{h,\tau}\|_m^2 + \sum_{i=1}^{6} I_i(t_n),$$

where

$$I_1(t_n) := 2\tau(v_n^{h,\tau}, \left(\tilde{\mathcal{L}}_{t_n}^h + I_\delta^h\right) v_n^{h,\tau})_m + 2\tau(v_{n-1}^{h,\tau}, \tilde{I}_{\delta^c}^h v_{n-1}^{h,\tau})_m + \|\eta(t_n)\|_m^2,$$
$$I_2(t_n) := 2\int_{]t_{n-1},t_n]} \sum_{i=0}^{d} (u_n^{h,\tau}, \delta_{h,i}F_t^i)_m dt,$$
$$I_3(t_n) := -\left\|\tau\left(\tilde{\mathcal{L}}_{t_n}^h + I_\delta^h\right) v_n^{h,\tau} + \sum_{i=0}^{d}\int_{[t_{n-1},t_n]} \delta_{h,i}F_t^i dt\right\|_m^2 + \left\|\tilde{I}_{\delta^c}^h v_{n-1}^{h,\tau}\right\|_m^2 \tau^2,$$
$$I_4(t_n) := 2\int_{]t_{n-1},t_n]} \left(v_{n-1}^{h,\tau}, \mathbf{1}_{n>1}\mathcal{N}_{t_{n-1}}^{\varrho;h} v_{n-1}^{h,\tau} + G_t^\varrho\right)_m dw_t^\varrho,$$
$$I_5(t_n) := 2\int_{]t_{n-1},t_n]}\int_{\mathbf{R}^d} \left(v_{n-1}^{h,\tau}, \mathbf{1}_{n>1}\mathcal{I}^h(z) v_{n-1}^{h,\tau} + R_t(z)\right)_m q(dt,dz),$$
$$I_6(t_n) := \left(\tau\tilde{I}_{\delta^c}^h v_{n-1}^{h,\tau}, \eta(t_n)\right)_m,$$

and where

$$\eta(t_n) := \int_{]t_{n-1},t_n]} \left(\mathbf{1}_{n>1}\mathcal{N}_{t_{n-1}}^{\varrho;h} v_{n-1}^{h,\tau} + G_t^\varrho\right) dw_t^\varrho + \int_{]t_{n-1},t_n]}\int_{\mathbf{R}^d} \left(\mathbf{1}_{n>1}\mathcal{I}^h(z) v_{n-1}^{h,\tau} + R_t(z)\right) q(dt,dz).$$

As in the proof Theorem 5.4.7, by Young's inequality, (5.4.10), and (5.4.15), we have

$$\begin{aligned}\mathbf{E}\left[\|v_n^{h,\tau}\|_m^2\right] \le{}& \left(1+2\pi^1(\{|z|>\delta\})\right)\mathbf{E}\left[\|\varphi\|_m^2\right] - \bar{q}_2\mathbf{E}\sum_{l=1}^{n}\tau\sum_{i=1}^{d}\|\delta_{h,i}v_l^{h,\tau}\|_m^2\\ &+\mathbf{E}\sum_{l=1}^{n}\tau\left(N_2+2\pi^1(\{|z|>\delta\})+\tau\pi^1(\{|z|>\delta\})^2\right)\|v_l^{h,\tau}\|_m^2\\ &+N\mathbf{E}\int_{]0,t_n]}\left(\sum_{i=0}^{d}\|F_t^i\|_m^2 dt + \|G_t\|_{m,\ell_2}^2 + \int_{\mathbf{R}^d}\|R_t(z)\|_m^2\pi^2(dz)\right)dt.\end{aligned}$$

Set

$$Z := N_2 + 2\pi^1(\{|z|>\delta\}),$$
$$R := \max\left(\frac{2\pi^1(\{|z|>\delta\})^2}{\sqrt{Z^2+4\pi^1(\{|z|>\delta\}^2} - Z}, N_2\right)T.$$

Assume $\mathcal{T} > R$. Making use of (5.4.29) and applying discrete Gronwall's lemma, we get that there exist a constant $N(d,m,K,\varkappa,T,\delta)$ such that

$$\mathbf{E}\left[\|v_n^{h,\tau}\|_m^2\right]+\mathbf{E}\sum_{l=1}^{n}\tau\sum_{i=0}^{d}\|\delta_{h,i}v_l^{h,\tau}\|_m^2\leq N\mathbf{E}\left[\|\varphi\|_m^2\right]$$

$$+N\mathbf{E}\int_{]0,T]}\left(\sum_{i=0}^{d}\|F_t^i\|_m^2+\|G_t\|_{m,\ell_2}^2+\int_{\mathbf{R}^d}\|R_t(z)\|_m^2\pi^2(dz)\right)dt. \tag{5.4.30}$$

Using (5.4.15) instead of (5.4.16), we obtain (5.4.22) from (5.4.30) in the same manner as Theorem 5.4.7. Note that no bound on τ/h^2 is needed in this case. □

5.5 Proof of the main theorems

Proof of Theorem 5.2.3. By virtue of Theorems 2.9, 2.10, and 4.1 in [Gyö82], in order to obtain the existence, uniqueness, regularity, and the estimate (5.2.4), we only need to show that (5.1.1) may be realized as an abstract stochastic evolution equation in a Gelfand triple and that the growth condition and coercivity condition are satisfied. Indeed, since (5.1.1) is a linear equation, the hemicontinuity condition is immediate and monotonicity follows directly from the coercivity condition. By Holder's inequality and Assumption 5.2.1(i), for $u,v\in H^1$, we have

$$\sum_{i,j=0}^{d}\left(\partial_j u,(v\partial_{-i}a_t^{ij}+a^{ij}\partial_{-i}v)\right)_0+\int_{|z|>\delta}\left(u(\cdot+z)-u-\mathbf{1}_{[-1,1]}(z)\sum_{j=1}^{d}z^j\partial_j u,v\right)_0\pi^1(dz)$$

$$+\int_{|z|\leq\delta}\int_0^1\sum_{i,j=1}^{d}\left(z^j\partial_j u(\cdot+\theta z),z^i\partial_{-i}v\right)_0(1-\theta)d\theta\pi^1(dz)\leq N\|u\|_1\|v\|_1.$$

Therefore, since the pairing $\langle\cdot,\cdot\rangle_1$ brings $(H^1)^*$ and H^{-1} into isomorphism, for each $(\omega,t)\in[0,T]\times\Omega$, there exists a linear operator $\tilde{A}_t:H^1\to H^{-1}$ such that $\langle v,\tilde{A}_t u\rangle_1$ agrees with the left-hand-side of the above inequality and for $u,v\in H^1$, $\|\tilde{A}_t u\|_{-1}\leq N\|u\|_1$. By Assumption 5.2.2, the operator A defined by $A(u)=\tilde{A}u+f$, maps H^1 to H^{-1} and for $u\in H^1$, $\|A_t(u)\|_{-1}\leq N(\|u\|_1+\|f\|_{-1})$.

For an integer $m\geq 1$, with abuse of notation, we write

$$(\cdot,\cdot)_m=((1-\Delta)^{m/2}\cdot,(1-\Delta)^{m/2}\cdot)_0.$$

and $\|\cdot\|_m$ for the corresponding norm in H^m. It is well known that the above inner product and norm are equivalent to the ones introduced in Section 1. For each $m\geq 1$ and for all $u\in H^{m+1}$ and $v\in H^m$, we have $(u,v)_m\leq\|u\|_{m+1}\|v\|_{m-1}$. Since H^{m+1} is dense in H^{m-1}, we may define the pairing $[\cdot,\cdot]_m:H^{m+1}\times H^{m-1}\to\mathbf{R}$ by $[v,v']_m=\lim_{n\to\infty}(v,v_n)_m$ for all

$v \in H^{m+1}$ and $v' \in H^{m-1}$, where $(v_n)_{n=1}^{\infty} \subset H^{m+1}$ is such that $\|v_n - v'\|_{m-1} \to 0$ as $n \to \infty$. It can be shown that the mapping from H^{m-1} to $(H^{m+1})^*$ given by $v' \mapsto [\cdot, v']_m$ is an isometric isomorphism. For more details, see [Roz90]. Therefore, for all $m \geq 0$, (H^{m+1}, H^m, H^{m-1}) forms a Gelfand triple with the pairing $[\cdot,\cdot]_m$, where we make the convention that $\langle \cdot,\cdot \rangle_1 = [\cdot,\cdot]_0$.

For $m \geq 1$ and all $u \in H^{m+1}$ and $v \in H^m$, using integration by parts, we get $\langle v, A_t(u)\rangle_1 = ((\mathcal{L}_t + I_t)u + f, v)_0 = \langle v, (\mathcal{L}_t + I_t)u + f\rangle_1$. Since this is true for all $v \in H^m$, which is dense in H^1, the restriction of A to H^{m+1} coincides with $L + I + f$. Moreover, it can easily be shown under Assumptions 5.2.1(i) and 5.2.2 that for all $m \geq 1$ and $u, v \in H^{m+1}$, $\|A_t(u)\|_{m-1} \leq N\|u\|_{m+1} + \|f\|_{m-1}$, where N is a constant depending only on m, d, K, and ν, which shows that A satisfies the growth condition. For $u \in H^m$, $m \geq 1$, define $B_t^{\varrho}(u) = b_t^{i\varrho}\partial_i u + g_t^{\varrho}$, $B_t = (B_t^{\varrho})_{\varrho=1}^{\infty}$, and $C_z(u) = u(\cdot + z) - u + o_t(z)$, $z \in \mathbf{R}^d$. Owing to Assumption 5.2.1 (i), B_t is an operator from H^{m+1} to $H^m(\ell_2)$. Furthermore, C is an operator from H^{m+1} to $L_2(\mathbf{R}^d, \pi^2(dz); H^m)$ (see (5.5.2)). It is also clear that A, B, and C are appropriately measurable. Thus, (5.1.1) may be realized as the following stochastic evolution equation in the Gelfand triple (H^{m+1}, H^m, H^{m-1}):

$$u_t = u_0 + \int_{]0,t]} A_s(u_s)ds + \int_{]0,t]} B_s^{\varrho}(u_s)dw_s^{\varrho} + \int_{]0,t]} C_z(u_{s-})q(dz, ds), \tag{5.5.1}$$

for $t \in [0,T]$. Let $u \in C_c^{\infty}$. A simple calculation shows that there is a constant $N = N(\delta)$ such that

$$\int_{\mathbf{R}^d} \|u(\cdot + z) - u\|_m^2 \pi^2(dz) \leq \varsigma_2(\delta)\|u\|_{m+1}^2 + N\|u\|_m^2. \tag{5.5.2}$$

Applying Holder's inequality and the identity $(u, \partial_j u) = 0$, we obtain

$$\int_{|z|>\delta'} \left(u(\cdot + z) - u - \mathbf{1}_{[-1,1]}(z)\sum_{j=1}^{d} z^j\partial_j u, u \right)_m \pi^1(dz) \leq 0.$$

By the Holder's inequality and the Cauchy-Bunyakovsky-Schwarz inequality, we have

$$2\int_{|z|\leq\delta'}\int_0^1 \sum_{i,j=1}^{d} \left(z^j\partial_j u(\cdot + \theta z), z^i\partial_{-i}u \right)_m (1-\theta)d\theta\pi^1(dz) \leq \varsigma_1(\delta)\|u\|_{m+1}^2.$$

There exists a constant $\epsilon = \epsilon(\varkappa, \delta)$ such that

$$\overline{q} := \varkappa - \varsigma(\delta) - \epsilon > 0.$$

As in Theorem 4.1.2 in [Roz90] and Lemma 5.4.4, using Holder's and Young's inequalities, the above estimates, and Assumption 5.2.1, we find that for each $\epsilon > 0$, there is a constant $N = N(d, m, \kappa, K, T, \delta)$ such that for all $(\omega, t) \in \Omega \times [0, T]$,

$$
\begin{aligned}
&2[u, A_t(u)]_m + \|B_t(u)\|^2_{m,l^2} + \int_{\mathbf{R}^d} \|C_z(u)\|^2_m \pi^2(dz) + \overline{q}\|u\|^2_{m+1} \\
&\leq N\left(\|u\|^2_m + \|f_t\|_{m-1} + \|g_t\|_{m,\ell_2} + \int_{\mathbf{R}^d} \|o_t(z)\|^2_m \pi^2(dz)\right).
\end{aligned}
$$

Using the self-adjointness of $(1-\Delta)^{1/2}$, the properties of the CBF $[\cdot,\cdot]_m$, and Assumption 5.2.2, for all $v \in C_c^\infty$ and $u \in H^{m+1}$, $m \geq 1$, we have

$$
[v, A(u)]_m = ((L + I)u, (1-\Delta)^m v)_0 + (f, (1-\Delta)^m v)_0. \tag{5.5.3}
$$

Owing to (5.5.3) and the denseness of $(1-\Delta)^{-m}C_c^\infty$ in H^1, from Theorems 2.9, 2.10, and 4.1 in [Gyö82], we obtain the existence and uniqueness of a solution u of (5.1.1), such that u is a càdlàg H^m-valued process satisfying (5.2.4). □

Proof of Proposition 5.2.5. Let A, B, and C be as in (5.5.1). Owing to Assumption 5.2.1, the boundedness of the $m-1$-norm of g in expectation, and estimate (5.2.4), using Jensen's inequality and Itô's isometry, for $s, t \in [0, T]$, we get

$$
\begin{aligned}
&\mathbf{E}\left[\left\|\int_{]s,t]} A_r(u_r)ds\right\|^2_{m-1}\right] \leq |t-s|\left(N\mathbf{E}\int_{]0,T]} \|u_t\|^2_{m+1}dt + \mathbf{E}\int_{]0,T]} \|f_r\|^2_{m-1}dr\right) \leq N|t-s|, \\
&\mathbf{E}\left[\left\|\int_{]s,t]} B^\varrho_r(u_r)dw^\varrho_r\right\|^2_{m-1}\right] = \mathbf{E}\int_{]s,t]} \|B_r(u_r)\|^2_{m-1,\ell_2}dr \\
&\leq N|t-s|\left(\sup_{t\leq T}\mathbf{E}\|u_t\|^2_m + \sup_{t\leq T}\mathbf{E}\|g_t\|_{m-1,\ell_2}\right) \leq N|t-s|,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{E}\left[\left\|\int_{]s,t]}\int_{\mathbf{R}^d} C_z(u_{r-})q(dr,dz)\right\|^2_{m-1}\right] = \mathbf{E}\int_{]s,t]}\int_{\mathbf{R}^d} \|C_z(u_r)\|^2_{m-1}\pi^2(dz)ds \\
&\leq N|t-s|\left(\sup_{t\leq T}\mathbf{E}\|u_t\|^2_m + \sup_{t\leq T}\mathbf{E}\int_{\mathbf{R}^d} \|o_t(z)\|^2_{m-1}\pi^2(dz)\right) \leq N|t-s|,
\end{aligned}
$$

which completes the proof of the proposition. □

Theorem 5.5.1. *Let Assumptions 5.2.1 through 5.2.5 hold for some $m \geq 2$. Let u be the solution of (5.1.1) and $(u_n^{h,\tau})_{n=0}^{\mathcal{T}}$ be defined by (5.4.1). Then there is a constant $N = N(d, m, \varkappa, K, T, C, \lambda, \kappa_m^2, \delta)$ such that*

$$
\mathbf{E}\left[\max_{0\leq n\leq\mathcal{T}} \|u_{t_n} - u_n^{h,\tau}\|^2_{m-2}\right] + \mathbf{E}\sum_{l=0}^{\mathcal{T}}\tau\sum_{i=0}^{d} \|\delta_{h,i}u_{t_l} - \delta_{h,i}u_l^{h,\tau}\|^2_{m-2}ds \leq N(\tau + |h|^2). \tag{5.5.4}
$$

Proof. For $t \in [0,T]$, let $\kappa_1(t) := t_{n-1}$ for $t \in]t_{n-1}, t_n]$, and set $e_n^{h,\tau} := u_n^{h,\tau} - u_{t_n}$. One can easily verify that $e_n^{h,\tau}$ satisfies in H^{m-2},

$$e_n^{h,\tau} = e_{n-1}^{h,\tau} + \int_{]t_{n-1},t_n]} \left((\mathcal{L}_{t_n-1}^h + I^h) e_{n-1}^{h,\tau} + \sum_{i=0}^d \delta_{h,i} F_t^i \right) dt + \int_{]t_{n-1},t_n]} \left(\mathcal{N}_{t_{n-1}}^{\varrho;h} e_{n-1}^{h,\tau} + G_t^\varrho \right) dw_t^\varrho$$
$$+ \int_{]t_{n-1},t_n]} \int_{\mathbf{R}^d} \left(\mathcal{I}^h(z) e_{n-1}^{h,\tau} + R_t(z) \right) q(dt,dz),$$

where

$$F_t^0 := (\mathcal{L}_{\kappa_1(t)}^h - \mathcal{L}_{\kappa_1(t)}) u_t + (\mathcal{L}_{\kappa_1(t)} - \mathcal{L}_t) u_t + (I^h - I) u_t + (f_{\kappa_1(t)} - f_t) + I_{\delta^c}^h (u_{\kappa_1(t)} - u_t)$$
$$+ \sum_{j=1}^d a_{\kappa_1(t)}^{0j} \delta_{-h,j} (u_{\kappa_1(t)} - u_t) + \sum_{i=0}^d a_{\kappa_1(t)}^{i0} \delta_{h,i} (u_{\kappa_1(t)} - u_t) - \sum_{i,j=1}^d \delta_{-h,j} (u_{\kappa_1(t)} - u_t)(\cdot + h) \delta_{h,i} a_{\kappa_1(t)}^{ij},$$

$$F_t^i := \sum_{j=1}^d a_{\kappa_1(t)}^{ij} \delta_{-h,j} (u_{\kappa_1(t)} - u_t) + \sum_{k=0}^{\infty} \sum_{l=1}^{\chi(h,k)} \bar{\theta}_l^{k,h} \zeta_{k,h}^{ij} \delta_{-h,j} (u_{\kappa_1(t)} - u_t)(\cdot + h z_{r_l^{h,k}})$$

$$G_t^\varrho : = (\mathcal{N}_{\kappa_1(t)}^\varrho - \mathcal{N}_t^\varrho) u_t + (\mathcal{N}_{\kappa_1(t)}^{\varrho;h} - \mathcal{N}_{\kappa_1(t)}^\varrho) u_t + \mathcal{N}_{\kappa_1(t)}^\varrho (u_{\kappa_1(t)} - u_t) + (g_{\kappa_1(t)}^\varrho - g_t^\varrho)$$

$$R_t^h(z) : = \left(\mathcal{I}^h(z) - \mathcal{I}(z) \right) u_{t-} + \mathcal{I}^h(z)(u_{\kappa_1(t)} - u_{t-}) + \left(o_{\kappa_1(t)}(z) - o_t(z) \right).$$

By Theorem 5.4.7, we have

$$\mathbf{E}\left[\max_{0 \le n \le \mathcal{T}} \|e_n^{h,\tau}\|_{m-2}^2 \right] + \mathbf{E} \sum_{n=0}^M \tau \sum_{i=0}^d \|\delta_{h,i} e_n^{h,\tau}\|_{m-2}^2$$

$$\le N \mathbf{E} \int_{]0,T]} \Big(\sum_{i=0}^d \|F_t^i\|_{m-2}^2 + \|G_t\|_{m-2,\ell_2}^2 + \int_{\mathbf{R}^d} \|R_t(z)\|_{m-2}^2 \pi^2(dz) \Big) dt.$$

Using Lemmas 5.4.1, 5.4.2, and 5.4.3 and Assumptions 5.2.1(i) and 5.2.4, the right-hand-side of the above relation can be estimated by

$$N \mathbf{E} \int_{]0,T]} \left(|h|^2 \|u_t\|_{m+1}^2 + |\kappa_1(t) - t| \|u_t\|_m^2 + \|u_{\kappa_1(t)} - u_t\|_{m-1}^2 \right) dt$$

$$+ N \mathbf{E} \int_{]0,T]} \left(\|f_{\kappa_1(t)} - f_t\|_{m-2}^2 + \|g_{\kappa_1(t)} - g_t\|_{m-2,\ell_2} + \int_{\mathbf{R}^d} \|o_{\kappa_1(t)}(z) - o_t(z)\|_{m-2} \pi^2(dz) \right) dt$$

where N depends only on $d, m, \varkappa, K, C, \lambda, T, \delta$ and ν. By virtue of (5.2.4), Proposition

5.2.5, and Assumption 5.2.3, we obtain (5.5.4), which completes the proof. □

Theorem 5.5.2. *Let Assumptions 5.2.1 through 5.2.4 hold with $m \geq 2$ and let u be the solution of (5.1.1). There exists a constant $R = R(d, m, \varkappa, K, \delta)$ such that if $\mathcal{T} > R$, then there exists a unique solution $(v^{h,\tau})_{n=0}^{\mathcal{T}}$ of (5.4.2) in H^{m-2}. Moreover, there is a constant $N = N(d, m, \varkappa, K, T, C, \lambda, \kappa_m^2, \delta)$ such that*

$$\mathbf{E}\left[\max_{0\leq n\leq\mathcal{T}} \|u_{t_n} - v_n^{h,\tau}\|_{m-2}^2\right] + \mathbf{E}\sum_{l=0}^{\mathcal{T}} \tau \sum_{i=0}^{d} \|\delta_{h,i}u_{t_l} - \delta_{h,i}v_l^{h,\tau}\|_{m-2}^2 ds \leq N(\tau + |h|^2). \tag{5.5.5}$$

Proof. The existence and uniqueness follows directly from Theorem 5.4.8. Let $\kappa_1(t)$ be as in the previous proof and set $\kappa_2(t) = t_n$ for $t \in]t_{n-1}, t_n]$. Let G and R be defined as in Theorem 5.5.1 and define $\bar{F}^i$ to be F^i with $\kappa_1(t)$ replaced with $\kappa_2(t)$. Set $e_n^{h,\tau} = v_n^{h,\tau} - u_{t_n}$. As in the proof of Theorem 5.5.1, we have

$$\begin{aligned} e_n^{h,\tau} = e_{n-1}^{h,\tau} &+ \int_{]t_{n-1},t_n]} \left((\tilde{\mathcal{L}}_{t_n}^h + I_\delta^h) e_n^{h,\tau} + \tilde{I}_{\delta^c}^h e_{n-1}^{h,\tau} + \sum_{i=0}^{d} \delta_{h,i}\tilde{F}_t^i \right) dt \\ &+ \int_{]t_{n-1},t_n]} \left(\mathbf{1}_{n>1} \mathcal{N}_{t_{n-1}}^{\varrho;h} e_{n-1}^{h,\tau} + \tilde{G}_t^\varrho \right) dw_t^\varrho + \int_{]t_{n-1},t_n]} \int_{\mathbf{R}^d} \left(\mathbf{1}_{n>1} \mathcal{I}^h(z) e_{n-1}^{h,\tau} + \tilde{R}_t(z) \right) q(dt, dz), \end{aligned}$$

where

$$\begin{aligned} \tilde{F}^i = \bar{F}^i, \text{ for } i \neq 0, \ \tilde{F}^0 &= \bar{F}^0 + \tilde{I}_{\delta^c}^h (u_{\kappa_1(t)} - u_{\kappa_2(t)}), \\ \tilde{G}_t^\varrho = \mathbf{1}_{t\leq t_1}(\mathcal{N}_t^\varrho u_t + g_t^\varrho) + \mathbf{1}_{t>t_1} G_t^\varrho, &\quad \tilde{R}_t(z) = \mathbf{1}_{t\leq t_1}\mathcal{I}(z)u_{t-} + \mathbf{1}_{t>t_1} R_t(z). \end{aligned}$$

By Theorem 5.4.8, we have

$$\mathbf{E}\left[\max_{0\leq n\leq\mathcal{T}} \|e_n^{h,\tau}\|_{m-2}^2\right] + \mathbf{E}\sum_{n=0}^{M} \tau \sum_{i=0}^{d} \|\delta_{h,i} e_n^{h,\tau}\|_{m-2}^2 \leq N(A_1 + A_2 + A_3),$$

where

$$\begin{aligned} A_1 &:= \mathbf{E}\int_{]0,T]} \sum_{i=0}^{d} \|\bar{F}_t^i\|_{m-2}^2 dt + \int_{]t_1,T]} \left(\|G_t\|_{m-2,\ell_2}^2 + \int_{\mathbf{R}^d} \|R_t(z)\|_{m-2}^2 \pi^2(dz) \right) dt, \\ A_2 &:= \mathbf{E}\int_{]0,T]} \|\tilde{I}_{\delta^c}^h (u_{\kappa_1(t)} - u_{\kappa_2(t)})\|_{m-2}^2 dt \\ A_3 &:= \mathbf{E}\int_{]0,t_1]} \left(\|M_t u_t + g_t\|_{m-2,\ell_2}^2 + \int_{\mathbf{R}^d} \|\mathcal{I}(z)u_t + o_t(z)\|_{m-2}^2 \pi^2(dz) \right) dt. \end{aligned}$$

As in the proof of Theorem 5.5.1, we have $A_1 \le N(\tau + |h|^2)$. By Proposition 5.2.5, we get

$$A_2 \le N\mathbf{E} \int_0^T ||u_{\kappa_1(t)} - u_{\kappa_2(t)}||_{m-1}^2 dt \le N\tau.$$

Owing to (5.2.3), we have

$$\begin{aligned} A_3 &\le N\mathbf{E} \int_0^{t_1} \left(||u_t||_{m-1}^2 + ||g_t||_{m-2,\ell_2}^2 + \int_{\mathbf{R}^d} ||o_t(z)||_{m-2}^2 \pi^2(dz) \right) dt \\ &\le N\tau\mathbf{E} \int_0^{t_1} \left(\sup_{t \le T} ||u_t||_{m-1}^2 + \xi \right) dt \le N\tau. \end{aligned}$$

Combining the above estimates yields (5.5.5). □

By virtue of Sobolev's embedding theorem and (5.2.10), as in [GK10], we obtain the following corollaries of Theorem 5.5.1 and Theorem 5.5.2.

Corollary 5.5.3. *Suppose the assumptions of Theorem 5.5.1 hold with $m > n + 2 + d/2$, where n is an integer with $n \ge 0$. Then for all $\lambda = (\lambda^1, \dots, \lambda^n) \in \{1 \dots, d\}^n$ and $\delta_{h,\lambda} = \delta_{h,\lambda^1} \cdots \delta_{h,\lambda^n}$, there is a constant $N = N(d, m, \varkappa, K, T, C, \lambda, \kappa_m^2, \delta)$ such that*

$$\mathbf{E}\left[\max_{0 \le n \le \mathcal{T}} \sup_{x \in \mathbf{R}^d} |\delta_{h,\lambda} u_{t_n}(x) - \delta_{h,\lambda} u_n^{h,\tau}(x)|^2 \right] + \mathbf{E}\left[\max_{0 \le n \le \mathcal{T}} ||\delta_{h,\lambda} u_{t_n} - \delta_{h,\lambda} u_n^{h,\tau}||_{\ell_2(\mathbf{G}_h)}^2 \right] \le N(\tau + |h|^2).$$

Corollary 5.5.4. *Suppose the assumptions of Theorem 5.5.2 hold with $m > n + 2 + d/2$, where n is an integer with $n \ge 0$. Then for all $\lambda = (\lambda^1, \dots, \lambda^n) \in \{1 \dots, d\}^n$ and $\delta_{h,\lambda} = \delta_{h,\lambda^1} \cdots \delta_{h,\lambda^n}$, there is a constant $N = N(d, m, \varkappa, K, T, C, \lambda, \kappa_m^2, \delta)$ such that*

$$\mathbf{E}\left[\max_{0 \le n \le \mathcal{T}} \sup_{x \in \mathbf{R}^d} |\delta_{h,\lambda} u_{t_n}(x) - \delta_{h,\lambda} v_n^{h,\tau}(x)|^2 \right] + \mathbf{E}\left[\max_{0 \le n \le \mathcal{T}} ||\delta_{h,\lambda} u_{t_n} - \delta_{h,\lambda} v_n^{h,\tau}||_{\ell_2(\mathbf{G}_h)}^2 \right] \le N(\tau + |h|^2).$$

Proof of Theorems 5.2.8 and 5.2.9. Let $(\hat{u}_n^{h,\tau})_{n=0}^M$ be defined by (5.2.8). Denote by $(\cdot, \cdot)_{\ell_2(\mathbf{G}_h)}$ the inner product of $\ell_2(\mathbf{G}_h)$. There exists a constant $\epsilon = \epsilon(\varkappa, \delta)$ such that

$$\overline{q} := \varkappa - \varsigma_1(\delta) - \epsilon > 0.$$

As in (5.4.11), there is a constant $N_6 = N_6(d, \varkappa, K, \delta)$ such that for all $\phi \in \ell_2(\mathbf{G}_h)$,

$$(\phi, \tilde{\mathcal{L}}_t^h \phi)_{\ell_2(\mathbf{G}_h)} + (\phi, I_\delta^h \phi)_{\ell_2(\mathbf{G}_h)} \le -\overline{q} \sum_{i=1}^d ||\delta_{h,i}\phi||_{\ell_2(\mathbf{G}_h)}^2 + N_6 ||\phi||_{\ell_2(\mathbf{G}_h)}^2.$$

Following the arguments in the beginning of the proof of Theorem 5.4.8, we conclude that if $\mathcal{T} > N_6 T$, then there exists a unique solution $(\hat{v}_n^{h,\tau})_{n=0}^M$ in $\ell_2(\mathbf{G}_h)$ of (5.2.9). It is easy to see that $N_6 < N_2$ (for the same choice of ϵ) for all $m > 0$, where N_2 is the constant appearing on the right-hand-side of (5.4.11), and hence $N_6 < R$, where R is as in Theorem 5.4.8. Let $(u_n^{h,\tau})_{n=1}^M$ be defined by (5.4.1). By Theorem 5.5.2, there exists a unique solution $(v_n^{h,\tau})_{n=1}^M$ of (5.4.2). It suffices to show that almost surely,

$$u_n^{h,\tau}(x) = \hat{u}_n^{h,\tau}(x) \tag{5.5.6}$$

and

$$v_n^{h,\tau}(x) = \hat{v}_n^{h,\tau}(x), \tag{5.5.7}$$

for all $n \in \{0, ..., M\}$ and $x \in \mathbf{G}^h$. Let $\mathscr{S} : H^{m-2} \to \ell_2(\mathbf{G}^h)$ denote the embedding from Remark 5.2.6. Applying $\mathscr{S}$ to both sides of (5.4.1), one can see that $\mathscr{S}u^{h,\tau}$ and $\hat{u}^{h,\tau}$ satisfy the same recursive relation in $\ell_2(\mathbf{G}^h)$ with common initial condition φ, and hence (5.5.6) follows. Similarly, $\mathscr{S}v^{h,\tau}$ and $\hat{v}^{h,\tau}$ satisfy the same equation in $\ell_2(\mathbf{G}^h)$ and (5.5.7) follows from the uniqueness of the $\ell_2(\mathbf{G}^h)$ solution of (5.2.9). □

Remark 5.5.5. It follows from Corollaries 5.5.3, 5.5.4, and relations (5.5.6) and (5.5.7) that if more regularity is assumed of the coefficients and the data of the equation (5.1.1), then better estimates can be obtained than the ones presented in Theorems 5.2.8 and 5.2.9.

Chapter 6
Conclusions and future work

In this thesis, we investigated existence, uniqueness, and regularity of degenerate parabolic linear SIDEs in the whole space with adapted coefficients from a couple of standpoints. First, we used the method of stochastic characteristics to derive classical solutions directly. Second, we derived the integer scale L^2-Sobolev theory for the equations using the variational framework of stochastic evolution equations and the method of vanishing viscosity. We then established the rate of convergence of some finite difference schemes for a simple class of SIDEs under the assumption of non-degenerate stochastic parabolicity.

There are myriad of future directions to consider. We will only discuss a few. As we mentioned in Chapter 4, the integer scale L^p-Sobolev theory for degenerate SIDEs is currently underway and will be available soon. Still though, it would be ideal to obtain an existence and regularity theory in the full scale of Bessel potential spaces with Hölder assumptions on the coefficients. It is also interesting to study non-degenerate linear SIDEs under weaker regularity conditions than those required for the fully degenerate theory. Currently, an existence theory is known only in a few special cases; we refer the reader to [MP09, MP11, MP12, MP13, KK12b, KK12a, KKK13] for some relatively recent results in this direction. With regards to approximations of SIDEs, we considered only equations with non-degenerate stochastic parabolicity and where the integral operators do not depend on the space variable. It would be interesting to relax these conditions in the future. Also, from a practical standpoint, the error from truncating the domain should be studied in conjunction with the numerical approximations of the equations. It is worth mentioning that a regularity theory for SIDEs in bounded domains with degenerate stochastic parabolicity is non-existent as far as the author knows. There is also no reason to confine oneself to finite difference schemes, since there are other numerical methods such as finite elements and wavelets. Although no simulations were done in this thesis, they will be considered in a future work along with some applications to the non-linear filtering problem.

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